

SENSITIVITIES VIA ROUGH PATHS

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ABSTRACT. Consider W a multidimensional centered and continuous Gaussian process with independent components such that a geometric rough path exists over it and X the solution (in rough paths sense) of a stochastic differential equation driven by W on $[0, T]$ with bounded coefficients ($T > 0$).

We prove the existence and compute the sensitivity of $\mathbb{E}[F(X_T)]$ to any variation of the initial condition and to any variation of the volatility function. On one hand, the theory of rough differential equations allows us to conclude when F is differentiable. On the other hand, using Malliavin calculus, the condition F is differentiable can be dropped under assumptions on the Cameron-Martin's space of W when $F \in L^2$.

Finally, we provide an application in finance in order to illustrate the link with the usual *computation of Greeks*.

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1. INTRODUCTION

Let W be a d -dimensional continuous and centered Gaussian process on $[0, T]$ with independent components ($d \in \mathbb{N}^*$ and $T > 0$).

Consider the stochastic differential equation (SDE) :

$$(1) \quad dX_t^{x,\sigma} = b(X_t^{x,\sigma}) dt + \sigma(X_t^{x,\sigma}) dW_t \text{ with } X_0^{x,\sigma} = x \in \mathbb{R}^d$$

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where b and σ are two bounded functions.

We show the existence and compute the sensitivity of

$$f_T(x, \sigma) = \mathbb{E}[F(X_T^{x, \sigma})]$$

to any variation of the initial condition x and then to any variation of the volatility function σ as well.

When W is a Brownian motion, it is well known that f_T is differentiable everywhere (cf. H. Kunita [19]). For every $x, v \in \mathbb{R}^d$, there exists a d -dimensional stochastic process $\tilde{h}^{x, v}$, adapted to the natural filtration of W on $[0, T]$, such that :

$$(2) \quad \partial_x f_T(x, \sigma) \cdot v = \mathbb{E}[F(X_T^{x, \sigma}) \delta(\tilde{h}^{x, v})]$$

where δ is the divergence operator matching with Itô's stochastic integral for processes adapted to the natural filtration of W . Similarly,

$$(3) \quad \partial_\sigma f_T(x, \sigma) \cdot \tilde{\sigma} = \mathbb{E}[F(X_T^{x, \sigma}) \delta(\tilde{\eta}^{\sigma, \tilde{\sigma}})]$$

where $\tilde{\eta}^{\sigma, \tilde{\sigma}}$ is a d -dimensional anticipative stochastic process defined on $[0, T]$.

In [10], E. Fournié et al. established (2) and (3) when W is a Brownian motion, b and σ are differentiable with bounded and Lipschitz derivatives and σ satisfies a uniform elliptic condition to ensure that $\tilde{h}^{x, v}$ and $\tilde{\eta}^{\sigma, \tilde{\sigma}}$ belongs to $\text{dom}(\delta)$. In [15], E. Gobet and R. Müños extended results of E. Fournié et al. [10] by supposing that σ only satisfies Hörmander's condition. For applications in Black-Scholes model and Vasicek's interest rate model, cf. [24], Chapter 2 and [29], Chapter 5. The case of signals with jumps is handled by N. Privault et al. in [18] and [31] but not covered here. Finally, J. Teichmann provides an estimator for weights $\delta(\tilde{h}^{x, v})$ and $\delta(\tilde{\eta}^{\sigma, \tilde{\sigma}})$ using cubature formulas when B is a Brownian motion (cf. J. Teichmann [37]).

The main purpose of this paper is to prove that (2) and (3) are still true when W is not a semimartingale. The deterministic rough paths framework dramatically simplifies every proofs, even in the Brownian motion's case mentioned above.

In order to take (1) as a rough differential equation (RDE), we will add sharper assumptions on W , b and σ in the sequel. Rough paths have been introduced by T. Lyons in [22]. Since 1998, many authors have developed that theory, in particular for stochastic analysis : P. Friz and N. Victoir, M. Gubinelli, A. Lejay, L. Coutin, S. Tindel, T. Lyons, etc. Here, the approach of P. Friz and N. Victoir is particularly well adapted because W is a Gaussian signal.

We also suggest applications of these results in finance. In an example, we consider a market defined by a SDE in which the volatility is the solution of an equation driven by a fractional Brownian motion. Then, we compute the sensitivity of an option's price to variations of that second equation's parameters. In that case, the rough paths approach is crucial and allows to go over limitations of the stochastic calculus framework.

At sections 2 and 3, we state useful results on rough differential equations (and extend some of them) coming from P. Friz and N. Victoir [11] and [12] and recently from T. Cass, C. Litterer and T. Lyons [2]. Section 4 (resp. 5) is devoted to prove the existence and compute the sensitivity of $f_T(x, \sigma)$ to variations of x (resp. σ) by using results of sections 2 and 3. The definition of the fractional Brownian motion

and some of its properties are provided at Section 6. At Section 7, we develop the application in finance mentioned above. Finally, at Section 8, we construct an estimator for each sensitivity when W is a fractional Brownian motion with Hurst parameter $H > 1/2$.

In order to take (1) as a rough differential equation, b and σ have to satisfy the following assumption :

Assumption 1.1. *There exists $p \geq 1$ such that :*

$$b \in C^{[p]+1}(\mathbb{R}^d) \text{ and } \sigma \in C^{[p]+1}[\mathbb{R}^d; \mathcal{M}(\mathbb{R}^d)].$$

Moreover, b and σ are bounded with bounded derivatives.

We denote by Σ the space of functions satisfying the same properties than σ , $\langle \cdot, \cdot \rangle$ the scalar product on \mathbb{R}^d , $\|\cdot\|$ the associated euclidean norm and $\|\cdot\|_{\mathcal{L}}$ (resp. $\|\cdot\|_{\mathcal{M}}$) the usual norm on $\mathcal{L}(\mathbb{R}^d)$ (resp. $\mathcal{M}_d(\mathbb{R})$).

In the sequel, we also assume that $F : \mathbb{R}^d \rightarrow \mathbb{R}$ satisfies one of the following hypothesis :

Assumption 1.2. *The function F belongs to $C^1(\mathbb{R}^d; \mathbb{R})$ and there exists $(C_F, N_F) \in \mathbb{R}_+^* \times \mathbb{N}^*$ such that :*

$$\forall x \in \mathbb{R}^d, |F(x)| \leq C_F(1 + \|x\|)^{N_F} \text{ and } \|DF(x)\|_{\mathcal{L}} \leq C_F(1 + \|x\|)^{N_F}.$$

Assumption 1.3. *The function F belongs to $L^2(\mathbb{R}^d)$ and there exists $(C_F, N_F) \in \mathbb{R}_+^* \times \mathbb{N}^*$ such that :*

$$\forall x \in \mathbb{R}^d, |F(x)| \leq C_F(1 + \|x\|)^{N_F}.$$

2. ROUGH DIFFERENTIAL EQUATIONS

In a sake of completeness, we present P. Friz and N. Victoir's approach of rough differential equations, [12], Part 2. Propositions 2.16, 2.17 and 2.19, and Lemma 2.18 are new (or extensions of existing) results.

For $0 \leq s < t \leq T$, consider $D_{s,t}$ the set of subdivisions for $[s, t]$,

$$\Delta_{s,t} = \{(u, v) \in \mathbb{R}_+^2 : s \leq u < v \leq t\} \text{ and } \Delta_T = \Delta_{0,T}.$$

Let $T^N(\mathbb{R}^d)$ be the step- N tensor algebra over \mathbb{R}^d ($N \in \mathbb{N}^*$) :

$$T^N(\mathbb{R}^d) = \bigoplus_{i=0}^N (\mathbb{R}^d)^{\otimes i}.$$

For $i = 1, \dots, N$, $(\mathbb{R}^d)^{\otimes i}$ is equipped with its euclidean norm $\|\cdot\|_i$.

Definition 2.1. *A function $\omega : \bar{\Delta}_T \rightarrow \mathbb{R}_+$ is a control if and only if, ω is continuous, $\omega(s, s) = 0$ for every $s \in [0, T]$ and ω is superadditive :*

$$\forall (s, t) \in \Delta_T, \forall u \in [s, t], \omega(s, u) + \omega(u, t) \leq \omega(s, t).$$

Definition 2.2. *For every $(s, t) \in \Delta_T$, a function $y : [s, t] \rightarrow \mathbb{R}^d$ is of finite p -variation if and only if,*

$$\|y\|_{p\text{-var}; s, t} = \sup_{D=\{r_k\} \in D_{s,t}} \left(\sum_{k=1}^{|D|-1} \|y_{r_{k+1}} - y_{r_k}\|^p \right)^{1/p} < \infty.$$

In the sequel, the space of continuous functions with finite p -variation will be denoted by :

$$C^{p\text{-var}}([s, t]; \mathbb{R}^d).$$

Definition 2.3. Let $y : [0, T] \rightarrow \mathbb{R}^d$ be a continuous function of finite 1-variation. The step- N signature of y is the functional $S_N(y) : \Delta_T \rightarrow T^N(\mathbb{R}^d)$ such that for every $(s, t) \in \Delta_T$ and $i = 1, \dots, N$,

$$S_{N;s,t}^i(y) = \int_{s < r_1 < r_2 < \dots < r_i < t} dy_{r_1} \otimes \dots \otimes dy_{r_i}.$$

Moreover,

$$G^N(\mathbb{R}^d) = \{S_{N;0,T}(y); y \in C^{1\text{-var}}([0, T]; \mathbb{R}^d)\}$$

is the step- N free nilpotent group over \mathbb{R}^d .

Definition 2.4. For every $(s, t) \in \Delta_T$, a map $Y : \Delta_{s,t} \rightarrow G^N(\mathbb{R}^d)$ is of finite p -variation if and only if,

$$\|Y\|_{p\text{-var};s,t} = \sup_{D=\{r_k\} \in D_{s,t}} \left(\sum_{k=1}^{|D|-1} \|Y_{r_k, r_{k+1}}\|_{\mathcal{C}}^p \right)^{1/p} < \infty$$

where, $\|\cdot\|_{\mathcal{C}}$ is the Carnot-Caratheodory's norm such that for every $g \in G^N(\mathbb{R}^d)$,

$$\|g\|_{\mathcal{C}} = \inf \{ \text{length}(y); y \in C^{1\text{-var}}([0, T]; \mathbb{R}^d) \text{ and } S_{N;0,T}(y) = g \}.$$

Remark. The Carnot-Caratheodory's norm induces a metric $d_{\mathcal{C}}$ on $G^N(\mathbb{R}^d)$, called Carnot-Caratheodory's distance (cf. [12], Section 7.5.4).

In the sequel, the space of continuous functions from $\Delta_{s,t}$ into $G^N(\mathbb{R}^d)$ with finite p -variation will be denoted by :

$$C^{p\text{-var}}([s, t]; G^N(\mathbb{R}^d)).$$

On that space, we consider the following metric called homogeneous distance in p -variation :

$$d_{p\text{-var};s,t}(X, Y) = \sup_{D=\{r_k\} \in D_{s,t}} \left[\sum_{k=1}^{|D|-1} d_{\mathcal{C}}^p(X_{r_k, r_{k+1}}, Y_{r_k, r_{k+1}}) \right]^{1/p}.$$

Let's define the Lipschitz regularity in the sense of Stein :

Definition 2.5. Consider $\gamma > 0$. A map $V : \mathbb{R}^d \rightarrow \mathbb{R}$ is γ -Lipschitz (in the sense of Stein) if and only if V is $C^{\lfloor \gamma \rfloor}$ on \mathbb{R}^d , bounded, with bounded derivatives and such that the $\lfloor \gamma \rfloor$ -th derivative of V is $\{\gamma\}$ -Hölder continuous ($\lfloor \gamma \rfloor$ is the largest integer strictly smaller than γ and $\{\gamma\} = \gamma - \lfloor \gamma \rfloor$).

Now, we remind the usual existence and uniqueness result of an ODE's solution :

Proposition 2.6. Consider $w : [0, T] \rightarrow \mathbb{R}^d$ a continuous function of finite 1-variation, V a collection of vector fields on \mathbb{R}^d and the ordinary differential equation :

$$(4) \quad dy_t = V(y_t) dw_t \text{ with initial condition } y_0 \in \mathbb{R}^d.$$

If V is continuous and bounded, (4) admits at least one solution. Moreover, if V is Lipschitz, (4) admits a unique solution denoted by $\pi_V(0, y_0; w)$.

The cornerstone of P. Friz and N. Victoir's results is Davie's lemma (cf. A.M. Davie [5]). Indeed, that lemma allows to extend Proposition 2.6 to the case of a function w of finite p -variation with $p > 1$.

It is stated as follow by P. Friz and N. Victoir (cf. [12], Lemma 10.7) :

Lemma 2.7. *Let V be a collection of $(\gamma - 1)$ -Lipschitz vector fields on \mathbb{R}^d ($\gamma > p$). There exists a constant $C_{DL} > 0$, depending on p and γ , such that for every $(s, t) \in \Delta_T$,*

$$\|\pi_V(0, y_0; w)\|_{p\text{-var}; s, t} \leq C_{DL} \times \left[\|V\|_{lip^{\gamma-1}} \|S_{[p]}(w)\|_{p\text{-var}; s, t} \vee \|V\|_{lip^{\gamma-1}}^p \|S_{[p]}(w)\|_{p\text{-var}; s, t}^p \right].$$

Now, w is just be a continuous function of finite p -variation such that a geometric p -rough path \mathbb{W} exists over it.

In other words, $\mathbb{W}_{s, t}^1 = w_t - w_s$ for every $(s, t) \in \Delta_T$ and, there exists an approximating sequence $(w^n, n \in \mathbb{N})$ of continuous functions of finite 1-variation such that :

$$\lim_{n \rightarrow \infty} d_{p\text{-var}; T} [S_{[p]}(w^n); \mathbb{W}] = 0.$$

Remark. By P. Friz and N. Victoir [12], Theorem 9.26, there also exists a geometric p -rough path over $\tilde{w} = (w, \text{Id}_{[0, T]})$:

$$\widetilde{\mathbb{W}} = S_{[p]}(\mathbb{W} \oplus \text{Id}_{[0, T]}).$$

It is useful in order to consider equations with a drift term. For a rigorous construction of Young pairing, the reader can refer to [12], Section 9.4.

Rigorously, a RDE's solution is defined as follow (cf. [12], Definition 10.17) :

Definition 2.8. *Let V be a collection of vector fields on \mathbb{R}^d . A continuous function $y : [0, T] \rightarrow \mathbb{R}^d$ is a solution of $dy = V(y)d\mathbb{W}$ with initial condition $y_0 \in \mathbb{R}^d$ if and only if,*

$$\lim_{n \rightarrow \infty} \|\pi_V(0, y_0; w^n) - y\|_{\infty; T} = 0$$

where, $\|\cdot\|_{\infty; T}$ is the uniform norm on $[0, T]$. If there exists a unique solution, it is denoted by $\pi_V(0, y_0; \mathbb{W})$.

Proposition 2.9. *Let V be a collection of $(\gamma - 1)$ -Lipschitz vector fields on \mathbb{R}^d ($\gamma > p$). Equation $dy = V(y)d\mathbb{W}$ with initial condition $y_0 \in \mathbb{R}^d$ admits at least one solution y (in the sense of Definition 2.8) and there exists a constant $C_{RDE} > 0$, depending on p and γ , such that for every $(s, t) \in \Delta_T$,*

$$(5) \quad \|y\|_{p\text{-var}; s, t} \leq C_{RDE} \left(\|V\|_{lip^{\gamma-1}} \|\mathbb{W}\|_{p\text{-var}; s, t} \vee \|V\|_{lip^{\gamma-1}}^p \|\mathbb{W}\|_{p\text{-var}; s, t}^p \right).$$

Moreover, if V is γ -Lipschitz, there exists a unique solution.

Remark. By reading carefully P. Friz and N. Victoir's proofs of [12], Proposition 10.3, Lemma 10.5, Lemma 10.7 and Theorem 10.14, one can show that C_{RDE} doesn't depend on y_0 , V and \mathbb{W} .

With the same ideas, P. Friz and N. Victoir proved similar results for full RDEs (cf. [12], Theorem 10.36) and RDEs driven along (affine-)linear vector fields (cf. [12], Theorem 10.53).

The notion of RDE's solution we defined above is matching with the notion of ODE's solution in rough paths sense of T. Lyons. Indeed, a RDE's solution for T.

Lyons, called a full RDE's solution by P. Friz and N. Victoir, must be a p -rough path (cf. [23], Section 6.3). Rigorously, a full RDE's solution is defined as follow (cf. [12], Definition 10.34) :

Definition 2.10. *A continuous function $\mathbb{Y} : \Delta_T \rightarrow G^{[p]}(\mathbb{R}^d)$ is a solution of the full rough differential equation $d\mathbb{Y} = V(\mathbb{Y})d\mathbb{W}$ with initial condition $\mathbb{Y}_0 \in G^{[p]}(\mathbb{R}^d)$ if and only if, $\mathbb{Y}_0 \otimes S_{[p]}(y^n)$ converges uniformly to \mathbb{Y} when $n \rightarrow \infty$, where $y^n = \pi_V(0, \mathbb{Y}_0^1; w^n)$. If there exists a unique solution, it is denoted by $\pi_V(0, \mathbb{Y}_0; \mathbb{W})$.*

The following proposition summarizes [12], Theorem 10.36 and Corollary 10.40 :

Proposition 2.11. *Let V be a collection of $(\gamma - 1)$ -Lipschitz vector fields on \mathbb{R}^d ($\gamma > p$). Equation $d\mathbb{Y} = V(\mathbb{Y})d\mathbb{W}$ with initial condition $\mathbb{Y}_0 \in G^{[p]}(\mathbb{R}^d)$ admits at least one solution \mathbb{Y} (in the sense of Definition 2.10) and there exists a constant $C_{FR} > 0$, depending on p and γ , such that for every $(s, t) \in \Delta_T$,*

$$\|\mathbb{Y}\|_{p\text{-var};s,t} \leq C_{FR} \left(\|V\|_{\text{lip}^{\gamma-1}} \|\mathbb{W}\|_{p\text{-var};s,t} \vee \|V\|_{\text{lip}^{\gamma-1}}^p \|\mathbb{W}\|_{p\text{-var};s,t}^p \right).$$

If V is γ -Lipschitz, there exists a unique solution.

Moreover, for every $R > 0$, the Itô map $(V, x, Y) \mapsto \pi_V(0, x; Y)$ is uniformly continuous from

$$\text{Lip}^\gamma(\mathbb{R}^d) \times \mathbb{R}^d \times \{\|Y\|_{p\text{-var};T} \leq R\} \text{ into } C^{p\text{-var}}([0, T]; G^{[p]}(\mathbb{R}^d)).$$

By equipping $\{\|Y\|_{p\text{-var};T} \leq R\}$ and $C^{p\text{-var}}([0, T]; G^{[p]}(\mathbb{R}^d))$ with $d_{\infty;T}$, the Itô map is still uniformly continuous.

When V is a collection of linear vector fields, we have the similar following result (cf. Theorem 10.53 and Exercice 10.55) :

Proposition 2.12. *Let V be the collection of linear vector fields defined by $V_i(y) = A_i y + b_i$ for every $y \in \mathbb{R}^d$ and $i = 1, \dots, d$ ($A_i \in \mathcal{M}_d(\mathbb{R})$ and $b_i \in \mathbb{R}^d$). Assume there exists a constant $M_{LR} > 0$ such that :*

$$\max_{i=1,\dots,d} \|A_i\|_{\mathcal{M}} + \|b_i\| \leq M_{LR}$$

and consider a control $\omega : \bar{\Delta}_T \rightarrow \mathbb{R}_+$ such that, for every $(s, t) \in \bar{\Delta}_T$,

$$M_{LR} \|\mathbb{W}\|_{p\text{-var};s,t} \leq \omega^{1/p}(s, t).$$

Equation $dy = V(y)d\mathbb{W}$ with initial condition $y_0 \in \mathbb{R}^d$ admits a unique solution and there exists a constant $C_{LR} > 0$, not depending on \mathbb{W} , such that :

$$\|\pi_V(0, y_0; \mathbb{W})\|_{\infty;T} \leq C_{LR}(1 + \|y_0\|) \exp \left[C_{LR} \sup_{\substack{D = \{r_k\} \in D_{0,T} \\ \omega(r_k, r_{k+1}) \leq 1}} \sum_{k=1}^{|D|-1} \omega(r_k, r_{k+1}) \right].$$

Moreover, there exists a constant $\tilde{C}_{LR} > 0$, not depending on \mathbb{W} , such that for every $(s, t) \in \Delta_T$,

$$\|\pi_{V;s,t}(0, y_0; \mathbb{W})\| \leq \tilde{C}_{LR}(1 + \|y_0\|) \omega^{1/p}(s, t) e^{\tilde{C}_{LR} \omega(0,T)}.$$

Remark. By the last inequality in the statement of Proposition 2.12, there exists a constant $C > 0$, not depending on \mathbb{W} , such that :

$$\|\pi_A(0, y_0; \mathbb{W})\|_{p\text{-var};T} \leq C e^{CR}$$

for any $R \geq \|\mathbb{W}\|_{p\text{-var};T}^p$.

Then, for $\tilde{A} \in \text{Lip}^\gamma(\mathbb{R}^d)$ such that $\tilde{A} \equiv A$ on the ball $\{\|u\| \leq C e^{CR}\}$,

$$\pi_A(0, y_0; \cdot) \equiv \pi_{\tilde{A}}(0, y_0; \cdot)$$

on the ball $\{\|Y\|_{p\text{-var};T}^p \leq R\}$.

Therefore, by Proposition 2.11 ; the map $Y \mapsto \pi_A(0, y_0; Y)$ is uniformly continuous from

$$\left\{ \|Y\|_{p\text{-var};T}^p \leq R \right\} \text{ into } C^{p\text{-var}}([0, T]; \mathbb{R}^d).$$

By equipping these sets with $d_{\infty;T}$, the map $Y \mapsto \pi_A(0, y_0; Y)$ is still uniformly continuous.

For P. Friz and N. Victoir, the rough integral of V along \mathbb{W} is the projection of a particular full RDE's solution (cf. [12], Definition 10.44) : $d\mathbb{Y} = \Phi(\mathbb{Y})d\mathbb{W}$ where,

$$\forall i = 1, \dots, d, \forall a, w \in \mathbb{R}^d, \Phi_i(w, a) = (e_i, V_i(w))$$

and (e_1, \dots, e_d) is the canonical basis of \mathbb{R}^d .

The following proposition ensures the existence, uniqueness and continuity of the rough integral when V is a collection of $(\gamma - 1)$ -Lipschitz vector fields (cf. [12], Theorem 10.47 and Corollary 10.48) :

Proposition 2.13. *Let V be a collection of $(\gamma - 1)$ -Lipschitz vector fields on \mathbb{R}^d ($\gamma > p$). There exists a unique rough integral of V along \mathbb{W} and there exists a constant $C_{RI} > 0$, depending on p and γ , such that for every $(s, t) \in \Delta_T$,*

$$\left\| \int V(\mathbb{W})d\mathbb{W} \right\|_{p\text{-var};s,t} \leq C_{RI} \|V\|_{\text{Lip}^{\gamma-1}} (\|\mathbb{W}\|_{p\text{-var};s,t} \vee \|\mathbb{W}\|_{p\text{-var};s,t}^p).$$

Moreover, for every $R > 0$, the rough integral

$$(V, Y) \mapsto \int V(Y)dY$$

is uniformly continuous from

$$\text{Lip}^{\gamma-1}(\mathbb{R}^d) \times \{\|Y\|_{p\text{-var};T} \leq R\} \text{ into } C^{p\text{-var}}([0, T]; G^{[p]}(\mathbb{R}^d)).$$

By equipping $\{\|Y\|_{p\text{-var};T} \leq R\}$ and $C^{p\text{-var}}([0, T]; G^{[p]}(\mathbb{R}^d))$ with $d_{\infty;T}$, the rough integral is still uniformly continuous.

Let's introduce some notations frequently used in the sequel and coming from [2] :

Notations. On one hand, for any $\alpha > 0$, any compact interval $I \subset \mathbb{R}_+$ and any control $\omega : \bar{\Delta}_T \rightarrow \mathbb{R}_+$,

$$M_{\alpha, I, \omega} = \sup_{\substack{D = \{r_k\} \in D_I \\ \omega(r_k, r_{k+1}) \leq \alpha}} \sum_{k=1}^{|D|-1} \omega(r_k, r_{k+1}).$$

In particular,

$$M_{\alpha, I, p}(\mathbb{W}) = M_{\alpha, I, \omega_{\mathbb{W}, p}}$$

where,

$$\forall (s, t) \in \bar{\Delta}_I, \omega_{\mathbb{W}, p}(s, t) = \|\mathbb{W}\|_{p\text{-var};s,t}^p.$$

On the other hand,

$$N_{\alpha, I, p}(\mathbb{W}) = \sup \{n \in \mathbb{N} : \tau_n \leq \sup(I)\}$$

where, for every $i \in \mathbb{N}$,

$$\begin{aligned} \tau_0 &= \inf(I) \text{ and} \\ \tau_{i+1} &= \inf \{t \in I : \|\mathbb{W}\|_{p\text{-var};\tau_i, t}^p \geq \alpha \text{ and } t > \tau_i\} \wedge \sup(I). \end{aligned}$$

Remarks :

- (1) Note that $\alpha \in \mathbb{R}_+ \mapsto M_{\alpha, I, p}(\mathbb{W})$ is increasing.
- (2) At [2], Proposition 4.6, T. Cass, C. Litterer and T. Lyons show that for every $\alpha > 0$,

$$(6) \quad M_{\alpha, I, p}(\mathbb{W}) \leq \alpha [2N_{\alpha, I, p}(\mathbb{W}) + 1].$$

In the sequel, $I = [0, T]$.

Lemma 2.14. *For every $q \in [1, p]$ satisfying $1/p + 1/q > 1$, there exists a constant $C(p, q) \geq 1$ such that :*

$$M_{\alpha, I, p}[S_{[p]}(\mathbb{W} \oplus h)] \leq C(p, q)[\|h\|_{q\text{-var}; T}^p + M_{\alpha, I, p}(\mathbb{W})]$$

for every $h \in C^{q\text{-var}}([0, T]; \mathbb{R}^d)$ and every $\alpha > 0$.

Proof. Consider $h \in C^{q\text{-var}}([0, T]; \mathbb{R}^d)$.

On one hand, for every $(s, t) \in \Delta_T$,

$$\omega_{\mathbb{W}, p}(s, t) = \|\mathbb{W}\|_{p\text{-var}; s, t} \leq \|S_{[p]}(\mathbb{W} \oplus h)\|_{p\text{-var}; s, t}.$$

On the other hand, since $\|\mathbb{W}\|_{p\text{-var}}^p$ and $\|h\|_{q\text{-var}}^q$ are controls, and $p/q \geq 1$:

$$\omega = \|\mathbb{W}\|_{p\text{-var}}^p + \|h\|_{q\text{-var}}^p = \omega_{\mathbb{W}, p} + (\|h\|_{q\text{-var}}^q)^{p/q}$$

is a control too.

Therefore, by [12], Theorem 9.26 and Exercice 9.21, there exists a constant $C(p, q) \geq 1$, not depending on h and \mathbb{W} , such that for every $(s, t) \in \Delta_T$,

$$\|S_{[p]}(\mathbb{W} \oplus h)\|_{p\text{-var}; s, t}^p \leq C(p, q)\omega(s, t).$$

In conclusion, for every $\alpha > 0$,

$$\begin{aligned} M_{\alpha, I, p}[S_{[p]}(\mathbb{W} \oplus h)] &\leq C(p, q) \sup_{\substack{D = \{r_k\} \in D_I \\ \omega_{\mathbb{W}, p}(r_k, r_{k+1}) \leq \alpha}} \sum_{k=1}^{|D|-1} \omega(r_k, r_{k+1}) \\ &\leq C(p, q) \left[\|h\|_{q\text{-var}; T}^p + M_{\alpha, I, p}(\mathbb{W}) \right] \end{aligned}$$

by super-additivity of the control $\|h\|_{q\text{-var}}^p$. \square

The following proposition provides a sharp upper bound for the Jacobian matrix of $\pi_V(0, \cdot; \mathbb{W})$. For the differentiability of that map cf. [12], Theorem 11.3, and for the upper bound cf. [2], Corollary 3.4 :

Proposition 2.15. *Let V be a collection of γ -Lipschitz vector fields ($\gamma > p$). The map $\pi_V(0, \cdot; \mathbb{W})$ is continuously differentiable on \mathbb{R}^d . For every $x \in \mathbb{R}^d$ and every $\alpha > 0$,*

$$(7) \quad \|J_{\cdot \leftarrow 0}^{x, \mathbb{W}}\|_{\infty; T} \leq C_{IC} e^{C_{IC} M_{\alpha, I, p}(\mathbb{W})}$$

where, $C_{IC} > 0$ is a constant depending only on p, γ, α and $\|V\|_{\text{lip}^\gamma}$, and the Jacobian matrix $J_{\cdot \leftarrow 0}^{x, \mathbb{W}}$ of $\pi_V(0, \cdot; \mathbb{W})$ at point x is viewed as a function belonging to $C^{p\text{-var}}([0, T], \mathbb{R}^{d^2})$.

Moreover, $J_{\cdot \leftarrow 0}^{x, \mathbb{W}}$ is a non singular matrix. For every $x \in \mathbb{R}^d$ and every $\alpha > 0$,

$$(8) \quad \|(J_{\cdot \leftarrow 0}^{x, \mathbb{W}})^{-1}\|_{\infty; T} \leq \tilde{C}_{IC} e^{\tilde{C}_{IC} M_{\alpha, I, p}(\mathbb{W})}$$

where, $\tilde{C}_{IC} > 0$ is a constant depending only on p, γ, α and $\|V\|_{\text{lip}^\gamma}$.

Remarks :

- (1) In the sequel, $J_{\cdot \leftarrow 0}^{x, \mathbb{W}}$ is simply denoted by $J_{\cdot \leftarrow 0}^{\mathbb{W}}$.

- (2) At [2], Corollary 3.4, authors only shown inequality (7). However, formally :

$$\begin{aligned} J_{t \leftarrow 0}^w &= I + \int_0^t \langle DV(y_s), J_{s \leftarrow 0}^w \rangle dw_s \text{ and} \\ (J_{t \leftarrow 0}^w)^{-1} &= I - \int_0^t \langle DV(y_s), (J_{s \leftarrow 0}^w)^{-1} \rangle dw_s. \end{aligned}$$

Then, one can show inequality (8) by replacing V by $-V$.

In the sequel, $(J_{\cdot \leftarrow 0}^w)^{-1}$ will be denoted by $J_{0 \leftarrow \cdot}^w$ and for every $(s, t) \in \Delta_T$, we put :

$$J_{s \leftarrow t}^w = J_{s \leftarrow 0}^w J_{0 \leftarrow t}^w \text{ and } J_{t \leftarrow s}^w = J_{t \leftarrow 0}^w J_{0 \leftarrow s}^w.$$

Then,

$$J_{s \leftarrow t}^w J_{t \leftarrow s}^w = J_{t \leftarrow s}^w J_{s \leftarrow t}^w = I.$$

- (3) By applying Lemma 2.14 to \mathbb{W} and $h = \text{Id}_{[0, T]}$:

$$M_{\alpha, I, p}(\widetilde{\mathbb{W}}) \leq C(p, 1) [T^p + M_{\alpha, I, p}(\mathbb{W})].$$

Then, from Proposition 2.15 (with its notation), for every $x \in \mathbb{R}^d$,

$$\begin{aligned} \|J_{\cdot \leftarrow 0}^{x, \widetilde{\mathbb{W}}}\|_{\infty; T} &\leq C_{\text{IC}} e^{C_{\text{IC}} M_{\alpha, I, p}(\widetilde{\mathbb{W}})} \\ &\leq \bar{C}_{\text{IC}} e^{\bar{C}_{\text{IC}} M_{\alpha, I, p}(\mathbb{W})} \end{aligned}$$

where,

$$\bar{C}_{\text{IC}} = C_{\text{IC}} \left[C(p, 1) \vee e^{C_{\text{IC}} C(p, 1) T^p} \right].$$

In order to use probabilistic results provided in [2] for equations with a drift term, that upper bound will be useful.

We now show that the Itô map is continuously differentiable with respect to the collection of vector fields and provide an upper bound similar to (7) for that derivative :

Proposition 2.16. *For every initial condition $y_0 \in \mathbb{R}^d$,*

$$V \in \text{Lip}^\gamma(\mathbb{R}^d) \mapsto y^{V, \mathbb{W}} = \pi_V(0, y_0; \mathbb{W})$$

is continuously differentiable ($\gamma > p$).

Moreover, for every $V, \tilde{V} \in \text{Lip}^\gamma(\mathbb{R}^d; \mathbb{R}^{d+1})$, there exists two constants $\alpha_{VF}(V, \tilde{V}) > 0$ and $C_{VF}(V, \tilde{V}) > 0$, not depending on \mathbb{W} , such that :

$$\|\partial_V y^{V, \tilde{\mathbb{W}}} \cdot \tilde{V}\|_{\infty; T} \leq C_{VF}(V, \tilde{V}) e^{C_{VF}(V, \tilde{V}) M_{\alpha_{VF}(V, \tilde{V}), I, p}(\mathbb{W})}.$$

Remark. Since proofs of propositions 2.11 and 2.13 follow the same pattern that the proof of Proposition 2.9 in [12], as the constant C_{RDE} ; C_{FR} and C_{RI} don't depend on y_0 , V and \mathbb{W} .

Proof. The first step of the proof follows the same pattern that P. Friz and N. Victoir's proof of [12], theorems 11.3 and 11.6. For every $V, \tilde{V} \in \text{Lip}^\gamma(\mathbb{R}^d)$, we construct a candidate for $\partial_V y^{V, \mathbb{W}} \cdot \tilde{V}$ by using an approximating sequence of \mathbb{W} . Then, we show that $y^{V, \mathbb{W}}$ is differentiable in the direction \tilde{V} by using Taylor's formula and continuity results of propositions 2.11 and 2.13. Finally, continuity results of propositions 2.11, 2.13 and 2.12 (cf. Remark) together with [12], Proposition B.5 allow to show that $V \mapsto \pi_V(0, y_0; \mathbb{W})$ is continuously differentiable on $\text{Lip}^\gamma(\mathbb{R}^d)$.

The second step of the proof is using similar ideas that in [12], Exercice 10.55.

Step 1. Since \mathbb{W} is a geometric p -rough path, there exists an approximating sequence $(w^n, n \in \mathbb{N})$ of continuous functions of finite 1-variation such that :

$$\lim_{n \rightarrow \infty} d_{p\text{-var}; T} [S_{[p]}(w^n); \mathbb{W}] = 0.$$

For every $n \in \mathbb{N}$ and $y_0 \in \mathbb{R}^d$,

$$V \in \text{Lip}^\gamma(\mathbb{R}^d) \mapsto y^{V;n} = \pi_V(0, y_0; w^n)$$

is continuously differentiable from the ODEs theory. Moreover, for every $V, \tilde{V} \in \text{Lip}^\gamma(\mathbb{R}^d)$ and every $t \in [0, T]$,

$$(9) \quad \partial_V y_t^{V;n} \cdot \tilde{V} = \int_0^t \langle DV(y_s^{V;n}), \partial_V y_s^{V;n} \cdot \tilde{V} \rangle dw_s^n + \int_0^t \tilde{V}(y_s^{V;n}) dw_s^n.$$

In order to obtain a candidate for $\partial_V y^{V;\mathbb{W}} \cdot \tilde{V}$, (9) has to be rewritten as follow :

$$d \left(\partial_V y_t^{V;n} \cdot \tilde{V} \right) = A \left(\partial_V y_t^{V;n} \cdot \tilde{V} \right) dz_t^{V, \tilde{V}; n}$$

with

$$dz_t^{V, \tilde{V}; n} = F_{V, \tilde{V}} \left(z_t^{V;n} \right) dz_t^{V;n} \text{ and } dz_t^{V;n} = F_V \left(z_t^{V;n} \right) dw_t^n$$

where, A , $F_{V, \tilde{V}}$ and F_V are three collections of vector fields such that for every $y, w, a_1, a_2 \in \mathbb{R}^d$ and $\Lambda \in \mathcal{L}(\mathbb{R}^d)$,

$$\begin{aligned} A(y) \cdot (\Lambda, w) &= \Lambda \cdot y + w, \\ F_{V, \tilde{V}}(y, a_1) \cdot (a_2, w) &= (\langle DV(y), \cdot \rangle w, \tilde{V}(y) \cdot w) \text{ and} \\ F_V(y) \cdot w &= (V(y) \cdot w, w). \end{aligned}$$

Then, by continuity results (with respect to the driving signal) for the Itô map and rough integral provided at propositions 2.11, 2.13 and 2.12 (cf. Remark) :

$$(10) \quad \partial_V y^{V;n} \cdot \tilde{V} = \varphi_n(V) \xrightarrow{n \rightarrow \infty} \varphi(V) \text{ in } (C^{p\text{-var}}([0, T]; \mathbb{R}^d); d_{\infty; T}),$$

with :

$$\begin{aligned} \varphi_n(V) &= \pi_A^1 \left[0, 0; \int F_{V, \tilde{V}}(Z^{V;n}) dZ^{V;n} \right] \text{ and} \\ (11) \quad \varphi(V) &= \pi_A^1 \left[0, 0; \int F_{V, \tilde{V}}(\mathbb{Z}^V) d\mathbb{Z}^V \right] \end{aligned}$$

where, for $\mathbb{Z}_0 = \exp[(y_0, 0)]$ (cf. [12], Chapter 7) :

$$Z^{V;n} = \pi_{F_V} [0, \mathbb{Z}_0; S_{[p]}(w^n)] \text{ and } \mathbb{Z}^V = \pi_{F_V}(0, \mathbb{Z}_0; \mathbb{W}).$$

Now, we show that $\partial_V y^{V;\mathbb{W}} \cdot \tilde{V}$ exists and is matching with $\varphi(V)$.

From Taylor's formula :

$$\pi_{V+\varepsilon \tilde{V}}(0, y_0; w^n) - \pi_V(0, y_0; w^n) = \int_0^\varepsilon \varphi_n(V + \theta \tilde{V}) d\theta$$

for every $\varepsilon \in [0, 1]$ and every $n \in \mathbb{N}$. Then, by Definition 2.8 :

$$(12) \quad \pi_{V+\varepsilon \tilde{V}}(0, y_0; \mathbb{W}) - \pi_V(0, y_0; \mathbb{W}) = \lim_{n \rightarrow \infty} \int_0^\varepsilon \varphi_n(V + \theta \tilde{V}) d\theta.$$

In order to permute limit/integration symbols in the right hand side of equality (12), it is sufficient to show that :

$$(13) \quad \sup_{n \in \mathbb{N}} \sup_{\theta \in [0, 1]} \left\| \varphi_n(V + \theta \tilde{V}) \right\|_{\infty; T} < \infty.$$

By propositions 2.13 and 2.11, for every $\theta \in [0, 1]$, $n \in \mathbb{N}$ and $(s, t) \in \Delta_T$, respectively :

$$(14) \quad C_{\text{RI}} \|F_{V+\theta\tilde{V}, \tilde{V}}\|_{\text{lip}^{\gamma-1}} \times \left(\|Z^{V+\theta\tilde{V}; n}\|_{p\text{-var}; s, t} \vee \|Z^{V+\theta\tilde{V}; n}\|_{p\text{-var}; s, t}^p \right) \leq \left\| \int F_{V+\theta\tilde{V}, \tilde{V}} (Z^{V+\theta\tilde{V}; n}) dZ^{V+\theta\tilde{V}; n} \right\|_{p\text{-var}; s, t}$$

and

$$(15) \quad \|Z^{V+\theta\tilde{V}; n}\|_{p\text{-var}; s, t} \leq C_{\text{FR}} [\|F_{V+\theta\tilde{V}}\|_{\text{lip}^{\gamma-1}} \|S_{[p]}(w^n)\|_{p\text{-var}; s, t} \vee \|F_{V+\theta\tilde{V}}\|_{\text{lip}^{\gamma-1}}^p \|S_{[p]}(w^n)\|_{p\text{-var}; s, t}^p].$$

From inequalities (14) and (15) :

$$\begin{aligned} \omega_{\theta, n}^{1/p}(s, t) &= \left\| \int F_{V+\theta\tilde{V}, \tilde{V}} (Z^{V+\theta\tilde{V}; n}) dZ^{V+\theta\tilde{V}; n} \right\|_{p\text{-var}; s, t} \\ &\leq \tilde{\omega}_n^{1/p}(s, t) \end{aligned}$$

where,

$$\tilde{\omega}_n^{1/p}(s, t) = \omega_n^{1/p}(s, t) \vee \omega_n(s, t) \vee \omega_n^p(s, t)$$

and

$$\omega_n^{1/p}(s, t) = \alpha^{1/p} \|S_{[p]}(w^n)\|_{p\text{-var}; s, t},$$

with :

$$\alpha^{1/p} = \sup_{\theta \in [0, 1]} \max_{k=1, p, p^2} \left[C_{\text{RI}} (C_{\text{FR}} \vee C_{\text{FR}}^p) \|F_{V+\theta\tilde{V}}\|_{\text{lip}^{\gamma-1}} \right]^{1/k} \|F_{V+\theta\tilde{V}}\|_{\text{lip}^{\gamma-1}} < \infty.$$

Then, by Proposition 2.12 :

$$\begin{aligned} \|\varphi_n(V + \theta\tilde{V})\|_{\infty; T} &\leq C_{\text{LR}} \exp \left[C_{\text{LR}} \sup_{\substack{D = \{r_k\} \in D_{0, T} \\ \tilde{\omega}_n(r_k, r_{k+1}) \leq 1}} \sum_{k=1}^{|D|-1} \tilde{\omega}_n(r_k, r_{k+1}) \right] \\ &= C_{\text{LR}} \exp \left[C_{\text{LR}} \sup_{\substack{D = \{r_k\} \in D_{0, T} \\ \omega_n(r_k, r_{k+1}) \leq 1}} \sum_{k=1}^{|D|-1} \omega_n(r_k, r_{k+1}) \right] \end{aligned}$$

because,

$$\tilde{\omega}_n \equiv \omega_n \text{ when } \tilde{\omega}_n \leq 1.$$

By super-additivity of ω_n :

$$\|\varphi_n(V + \theta\tilde{V})\|_{\infty; T} \leq C_{\text{LR}} e^{\alpha C_{\text{LR}}} \|S_{[p]}(w^n)\|_{p\text{-var}; T}^p.$$

Since the right hand side of the previous inequality does not depend on θ , and

$$\sup_{n \in \mathbb{N}} \|S_{[p]}(w^n)\|_{p\text{-var}; T}^p < \infty$$

by construction ; (13) is true.

As mentioned above, (10), (12) and (13) together imply via Lebesgue's theorem that :

$$\pi_{V+\varepsilon\tilde{V}}(0, y_0; \mathbb{W}) - \pi_V(0, y_0; \mathbb{W}) = \int_0^\varepsilon \varphi(V + \theta\tilde{V}) d\theta.$$

Moreover, by continuity results for the Itô map and rough integral provided at propositions 2.11, 2.13 and 2.12 (cf. Remark) ; $\varphi(V + \cdot \tilde{V})$ is continuous from $[0, 1]$ into $C^{p\text{-var}}([0, T]; \mathbb{R}^d)$.

Then, from [12], Proposition B.1 (Banach calculus), $\partial_V y^{V, \mathbb{W}} \cdot \tilde{V}$ exists and is matching with $\varphi(V)$.

Finally, $(V, \tilde{V}, w) \mapsto F_{V, \tilde{V}}(w)$ is uniformly continuous on bounded sets of

$$\text{Lip}^\gamma(\mathbb{R}^d) \times \text{Lip}^\gamma(\mathbb{R}^d) \times \mathbb{R}^d \times \mathbb{R}^d$$

by construction. Then, by propositions 2.11, 2.13 and 2.12 (cf. Remark) together with expression (11) of the directional derivative ; $(V, \tilde{V}) \mapsto \partial_V y^{V, \mathbb{W}} \cdot \tilde{V}$ is uniformly continuous on bounded sets of

$$\text{Lip}^\gamma(\mathbb{R}^d) \times \text{Lip}^\gamma(\mathbb{R}^d).$$

In conclusion, by [12], Proposition B.5 (Banach calculus), $V \mapsto \pi_V(0, y_0; \mathbb{W})$ is continuously differentiable on $\text{Lip}^\gamma(\mathbb{R}^d)$.

Step 2. Consider $V, \tilde{V} \in \text{Lip}^\gamma(\mathbb{R}^d; \mathbb{R}^{d+1})$. The first step is still true by replacing \mathbb{W} by $\widetilde{\mathbb{W}}$ with these new collections of vector fields.

By propositions 2.13 and 2.11, for every $(s, t) \in \Delta_T$, respectively :

$$(16) \quad \left\| \int_{F_{V, \tilde{V}}} (\mathbb{Z}^V) d\mathbb{Z}^V \right\|_{p\text{-var}; s, t} \leq C_{\text{RI}} \|F_{V, \tilde{V}}\|_{\text{lip}^{\gamma-1}} \times (\|\mathbb{Z}^V\|_{p\text{-var}; s, t} \vee \|\mathbb{Z}^V\|_{p\text{-var}; s, t}^p)$$

and

$$(17) \quad \|\mathbb{Z}^V\|_{p\text{-var}; s, t} \leq C_{\text{FR}} \left(\|F_V\|_{\text{lip}^{\gamma-1}} \|\widetilde{\mathbb{W}}\|_{p\text{-var}; s, t} \vee \|F_V\|_{\text{lip}^{\gamma-1}}^p \|\widetilde{\mathbb{W}}\|_{p\text{-var}; s, t}^p \right).$$

On one hand, from inequalities (16) and (17) :

$$\begin{aligned} \omega^{1/p}(s, t) &= \left\| \int_{F_{V, \tilde{V}}} (\mathbb{Z}^V) d\mathbb{Z}^V \right\|_{p\text{-var}; s, t} \\ &\leq \tilde{\omega}_0^{1/p}(s, t) \end{aligned}$$

where,

$$\tilde{\omega}_0^{1/p}(s, t) = \omega_0^{1/p}(s, t) \vee \omega_0(s, t) \vee \omega_0^p(s, t)$$

and

$$\omega_0^{1/p}(s, t) = \alpha_0^{1/p}(V, \tilde{V}) \|\widetilde{\mathbb{W}}\|_{p\text{-var}; s, t},$$

with :

$$\alpha_0^{1/p}(V, \tilde{V}) = \max_{k=1, p, p^2} \left[C_{\text{RI}} (C_{\text{FR}} \vee C_{\text{FR}}^p) \|F_{V, \tilde{V}}\|_{\text{lip}^{\gamma-1}} \right]^{1/k} \|F_V\|_{\text{lip}^{\gamma-1}}.$$

On the other hand, by Proposition 2.12 :

$$\begin{aligned} \|\partial_V y^{V, \widetilde{\mathbb{W}}} \cdot \tilde{V}\|_{\infty; T} &\leq C_{\text{LR}} \exp \left[C_{\text{LR}} \sup_{\substack{D = \{r_k\} \in D_{0, T} \\ \tilde{\omega}_0(r_k, r_{k+1}) \leq 1}} \sum_{k=1}^{|D|-1} \tilde{\omega}_0(r_k, r_{k+1}) \right] \\ &= C_{\text{LR}} \exp \left[C_{\text{LR}} \sup_{\substack{D = \{r_k\} \in D_{0, T} \\ \omega_0(r_k, r_{k+1}) \leq 1}} \sum_{k=1}^{|D|-1} \omega_0(r_k, r_{k+1}) \right] \end{aligned}$$

because,

$$\tilde{\omega}_0 \equiv \omega_0 \text{ when } \tilde{\omega}_0 \leq 1.$$

With notations of [2] :

$$\begin{aligned} \sup_{\substack{D = \{r_k\} \in D_{0,T} \\ \omega_0(r_k, r_{k+1}) \leq 1}} \sum_{k=1}^{|D|-1} \omega_0(r_k, r_{k+1}) &= \alpha_0(V, \tilde{V}) M_{\alpha_0^{-1}(V, \tilde{V}), I, p}(\widetilde{\mathbb{W}}) \\ &\leq C(1, p) \alpha_0(V, \tilde{V}) \left[T^p + M_{\alpha_0^{-1}(V, \tilde{V}), I, p}(\mathbb{W}) \right] \end{aligned}$$

by Lemma 2.14 applied to \mathbb{W} and $h = \text{Id}_{[0, T]}$.

In conclusion,

$$\|\partial_V y^{V, \widetilde{\mathbb{W}}} \cdot \tilde{V}\|_{\infty; T} \leq C_{\text{VF}}(V, \tilde{V}) e^{C_{\text{VF}}(V, \tilde{V}) M_{\alpha_{\text{VF}}(V, \tilde{V}), I, p}(\mathbb{W})}$$

where,

$$\alpha_{\text{VF}}(V, \tilde{V}) = \alpha_0^{-1}(V, \tilde{V})$$

and

$$C_{\text{VF}}(V, \tilde{V}) = C_{\text{LR}} \left[C(1, p) \alpha_0(V, \tilde{V}) + e^{C_{\text{LR}} C(1, p) \alpha_0(V, \tilde{V}) T^p} \right].$$

□

Remarks :

- (1) At step 1, since $F_{V+\theta\tilde{V}, \tilde{V}}$ involves $DV + \theta D\tilde{V}$ for $\theta \in [0, 1]$, it is necessary to assume that $V, \tilde{V} \in \text{Lip}^\gamma(\mathbb{R}^d)$ to get :

$$F_{V+\theta\tilde{V}, \tilde{V}} \in \text{Lip}^{\gamma-1}(\mathbb{R}^d) \text{ and } F_{V+\theta\tilde{V}} \in \text{Lip}^\gamma(\mathbb{R}^d)$$

in order to apply propositions 2.11 and 2.13.

- (2) C_{LR} is not depending on V, \tilde{V} and \mathbb{W} , because only the driving signal

$$\int F_{V, \tilde{V}}(\mathbb{Z}^V) d\mathbb{Z}^V$$

depends on them ; not the collection of linear vector fields A .

Finally, in order to work with Malliavin calculus, we need some results on Itô map's differentiability with respect to the driving signal.

Remark. In the sequel, the translation operator T_h on $C^{p\text{-var}}([0, T]; G^{[p]}(\mathbb{R}^d))$ for $h \in C^{q\text{-var}}([0, T]; \mathbb{R}^d)$ with $q \in [1, p]$ and $1/p + 1/q > 1$, will be used several times. It is naturally defined by $T_h S_{[p]}(w) = S_{[p]}(w + h)$ when $h, w \in C^{1\text{-var}}([0, T]; \mathbb{R}^d)$. For a rigorous construction of T_h , the reader can refer to [12], Section 9.4.6.

Proposition 2.17. *Consider $q \in [1, p]$ such that $1/p + 1/q > 1$. For every initial condition $y_0 \in \mathbb{R}^d$ and every collection V of γ -Lipschitz vector fields on \mathbb{R}^d ($\gamma > p$),*

$$\vartheta^{\mathbb{W}} : \begin{cases} C^{q\text{-var}}([0, T]; \mathbb{R}^d) \longrightarrow C^{p\text{-var}}([0, T]; \mathbb{R}^d) \\ h \longmapsto \pi_V(0, y_0; T_h \mathbb{W}) \end{cases}$$

is continuously differentiable.

Moreover, consider a control $\omega : \bar{\Delta}_T \rightarrow \mathbb{R}_+$ such that :

$$\forall (s, t) \in \bar{\Delta}_T, \|\mathbb{W}\|_{p\text{-var}; s, t}^p \leq \omega(s, t).$$

There exists a constant $C_\vartheta > 0$, not depending on \mathbb{W} , ω and $h \in C^{q\text{-var}}([0, T]; \mathbb{R}^d)$, such that :

$$\|D_h \vartheta^{\mathbb{W}}\|_{\infty; T} \leq C_\vartheta e^{C_\vartheta (\|h\|_{q\text{-var}; T}^p + M_{1, I, \omega})}$$

where,

$$D_h \vartheta^{\mathbb{W}} = \left\{ \frac{d}{d\varepsilon} \vartheta^{\mathbb{W}}(\varepsilon h) \right\}_{\varepsilon=0}.$$

Remark. When $q = p \in [1, 2]$, $T_{\varepsilon h} \mathbb{W} = w + \varepsilon h$. Then, $D_h \vartheta^w = \partial_w \vartheta^w(0).h$. In that case, $\vartheta^w(0)$ will be naturally denoted by $\vartheta(w)$ in the sequel.

Proof. By [12], Theorem 11.6, $\vartheta^{\mathbb{W}}$ is continuously differentiable.

Consider $h \in C^{q\text{-var}}([0, T]; \mathbb{R}^d)$.

By putting $\mathbb{Y}_0 = \exp[(y_0, 0, 0)]$, from [12], Theorem 11.3 :

$$D_h \vartheta^{\mathbb{W}} = \pi_A^1 \left[0, 0; \int F [\pi_G [0, \mathbb{Y}_0; S_{[p]}(\mathbb{W} \oplus h)]] d\pi_G [0, \mathbb{Y}_0; S_{[p]}(\mathbb{W} \oplus h)] \right]$$

where A, G and F are three collections of vector fields such that for every $\Lambda \in \mathcal{L}(\mathbb{R}^d)$ and every $y, w, h, a_1, a_2, a_3 \in \mathbb{R}^d$,

$$\begin{aligned} A(y).(\Lambda, w) &= \Lambda.y + w, \\ F(y, a_1, a_2).(a_3, w, h) &= (\langle DV(y), \cdot \rangle w, V(y).h) \text{ and} \\ G(y).(w, h) &= (V(y).w, w, h). \end{aligned}$$

By propositions 2.13 and 2.11, for every $(s, t) \in \Delta_T$,

$$\begin{aligned} \tilde{\omega}^{1/p}(s, t) &= \left\| \int F [\pi_G [0, \mathbb{Y}_0; S_{[p]}(\mathbb{W} \oplus h)]] d\pi_G [0, \mathbb{Y}_0; S_{[p]}(\mathbb{W} \oplus h)] \right\|_{p\text{-var}; s, t} \\ &\leq \tilde{\omega}_0^{1/p}(s, t) \end{aligned}$$

with

$$\tilde{\omega}_0^{1/p}(s, t) = \omega_0^{1/p}(s, t) \vee \omega_0(s, t) \vee \omega_0^p(s, t)$$

and, by [12], Theorem 9.26 and Exercice 9.21 :

$$\begin{aligned} \omega_0(s, t) &= \alpha_0 [\|h\|_{q\text{-var}; s, t}^p + \omega(s, t)] \\ &\geq \alpha_1 \|S_{[p]}(\mathbb{W} \oplus h)\|_{p\text{-var}; s, t}^p \end{aligned}$$

where, $\alpha_0, \alpha_1 \geq 1$ are two constants not depending on \mathbb{W} , ω and h .

Then, by Proposition 2.12 :

$$\begin{aligned} \|D_h \vartheta\|_{\infty; T} &\leq C_{\text{LR}} \exp \left[C_{\text{LR}} \sup_{\substack{D = \{r_k\} \in D_{0, T} \\ \tilde{\omega}_0(r_k, r_{k+1}) \leq 1}} \sum_{k=1}^{|D|-1} \tilde{\omega}_0(r_k, r_{k+1}) \right] \\ &= C_{\text{LR}} \exp \left[C_{\text{LR}} \sup_{\substack{D = \{r_k\} \in D_{0, T} \\ \omega_0(r_k, r_{k+1}) \leq 1}} \sum_{k=1}^{|D|-1} \omega_0(r_k, r_{k+1}) \right] \\ &\leq C_{\vartheta} \exp \left[C_{\vartheta} \left[\|h\|_{q\text{-var}; T}^p + \sup_{\substack{D = \{r_k\} \in D_{0, T} \\ \omega(r_k, r_{k+1}) \leq 1}} \sum_{k=1}^{|D|-1} \omega(r_k, r_{k+1}) \right] \right] \end{aligned}$$

because,

$$\tilde{\omega}_0 \equiv \omega_0 \text{ when } \tilde{\omega}_0 \leq 1,$$

and for every $(s, t) \in \Delta_T$, $\omega(s, t) \leq \omega_0(s, t)$. \square

Lemma 2.18. Consider $p \in [1, 2]$, and let A be the collection of linear vector fields defined by $A(y).(\Lambda, w) = \Lambda.y + w$ for every $y, w \in \mathbb{R}^d$ and every $\Lambda \in \mathcal{L}(\mathbb{R}^d)$. For every initial condition $y_0 \in \mathbb{R}^d$,

$$\Theta : \begin{cases} C^{p\text{-var}}([0, T]; \mathcal{L}(\mathbb{R}^d) \times \mathbb{R}^d) \longrightarrow C^{p\text{-var}}([0, T]; \mathbb{R}^d) \\ (\Lambda, w) \longmapsto \pi_A [0, y_0; (\Lambda, w)] \end{cases}$$

is continuously differentiable.

Moreover, consider a control $\omega : \bar{\Delta}_T \rightarrow \mathbb{R}_+$ such that :

$$\forall (s, t) \in \bar{\Delta}_T, \quad \|(\Lambda, w)\|_{p\text{-var}; s, t}^p \leq \omega(s, t).$$

There exists a constant $C_\Theta > 0$, not depending on $\tilde{w} = (\Lambda, w)$, ω and $\tilde{h} \in C^{p\text{-var}}([0, T]; \mathcal{L}(\mathbb{R}^d) \times \mathbb{R}^d)$, such that :

$$\left\| \partial_{\tilde{w}} \Theta(\tilde{w}).\tilde{h} \right\|_{\infty; T} \leq C_\Theta (1 + \|y_0\|) e^{C_\Theta (\|\tilde{h}\|_{p\text{-var}; T}^p + M_{1, T, \omega})}.$$

Proof. Since A is a collection of linear vector fields, by [12], Theorem 11.3 ; Θ is derivable at every points and in every directions on $C^{p\text{-var}}([0, T]; \mathcal{L}(\mathbb{R}^d) \times \mathbb{R}^d)$.

Consider $\tilde{h} = (H, h)$ belonging to $C^{p\text{-var}}([0, T]; \mathcal{L}(\mathbb{R}^d) \times \mathbb{R}^d)$.

On one hand, formally :

$$\begin{aligned} \partial_{\tilde{w}} \Theta(\tilde{w}).\tilde{h} &= \int_0^\cdot d\Lambda_s. \langle \partial_{\tilde{w}} \Theta_s(\tilde{w}), \tilde{h} \rangle + \int_0^\cdot dH_s. \Theta_s(\tilde{w}) + h \\ &= \int_0^\cdot \tilde{A} \left[\Theta_s(\tilde{w}); \partial_{\tilde{w}} \Theta_s(\tilde{w}).\tilde{h} \right] (d\tilde{w}_s, d\tilde{h}_s) \end{aligned}$$

where, \tilde{A} is the collection of linear vector fields defined by :

$$\tilde{A}(x, y).(w, h) = w^1.y + h^1.x + h^2$$

for every $x, y \in \mathbb{R}^d$ and $w, h \in \mathcal{L}(\mathbb{R}^d) \times \mathbb{R}^d$.

By putting $\Phi(\tilde{w}, \tilde{h}) = (\Theta(\tilde{w}); \partial_{\tilde{w}} \Theta(\tilde{w}).\tilde{h})$, still formally :

$$\Phi(\tilde{w}, \tilde{h}) = (y_0, 0) + \int_0^\cdot \bar{A} \left[\Phi_s(\tilde{w}, \tilde{h}) \right] (d\tilde{w}_s, d\tilde{h}_s)$$

where, \bar{A} is the collection of linear vector fields defined by :

$$\begin{aligned} \bar{A}(x, y).(w, h) &= \begin{bmatrix} A(x).w \\ \tilde{A}(x, y).(w, h) \end{bmatrix} \\ &= \begin{pmatrix} 0 \\ w^1.y \end{pmatrix} + \begin{pmatrix} w^1.x \\ h^1.x \end{pmatrix} + \begin{pmatrix} 0 \\ h^2 \end{pmatrix} \end{aligned}$$

for every $x, y \in \mathbb{R}^d$ and $w, h \in \mathcal{L}(\mathbb{R}^d) \times \mathbb{R}^d$.

Therefore, rigorously :

$$\Phi(\tilde{w}, \tilde{h}) = \pi_{\bar{A}} \left[0, (y_0, 0); (\tilde{w}, \tilde{h}) \right].$$

On the other hand, let $\tilde{\omega} : \bar{\Delta}_T \rightarrow \mathbb{R}_+$ be the control such that, for every $(s, t) \in \Delta_T$,

$$\begin{aligned} \tilde{\omega}(s, t) &= \alpha \left[\|\tilde{h}\|_{p\text{-var}; s, t}^p + \omega(s, t) \right] \\ &\geq \|(\tilde{w}, \tilde{h})\|_{p\text{-var}; s, t}^p \end{aligned}$$

where, $\alpha \geq 1$ is a constant not depending on \tilde{w} , ω and \tilde{h} .

By Proposition 2.12 :

$$\begin{aligned}
\left\| \partial_{\tilde{w}} \Theta(\tilde{w}).\tilde{h} \right\|_{\infty;T} &\leq \left\| \Phi(\tilde{w}, \tilde{h}) \right\|_{\infty;T} \\
&\leq C_{\text{LR}}(1 + \|y_0\|) \exp \left[C_{\text{LR}} \sup_{\substack{D = \{r_k\} \in D_{0,T} \\ \tilde{\omega}(r_k, r_{k+1}) \leq 1}} \sum_{k=1}^{|D|-1} \tilde{\omega}(r_k, r_{k+1}) \right] \\
&\leq C_{\Theta}(1 + \|y_0\|) \times \\
&\quad \exp \left[C_{\Theta} \left[\|\tilde{h}\|_{p\text{-var};T}^p + \sup_{\substack{D = \{r_k\} \in D_{0,T} \\ \omega(r_k, r_{k+1}) \leq 1}} \sum_{k=1}^{|D|-1} \omega(r_k, r_{k+1}) \right] \right]
\end{aligned}$$

because, for every $(s, t) \in \Delta_T$, $\omega(s, t) \leq \tilde{\omega}(s, t)$.

Finally, by Proposition 2.12 (cf. Remark), since $\Phi = \pi_{\bar{A}}[0, (y_0, 0); \cdot]$ and \bar{A} is a collection of linear vector fields ; Φ and then $(\tilde{w}, \tilde{h}) \mapsto \partial_{\tilde{w}} \Theta(\tilde{w}).\tilde{h}$ are uniformly continuous on bounded sets of

$$C^{p\text{-var}}([0, T]; \mathcal{L}(\mathbb{R}^d) \times \mathbb{R}^d) \times C^{p\text{-var}}([0, T]; \mathcal{L}(\mathbb{R}^d) \times \mathbb{R}^d).$$

Therefore, by [12], Proposition B.5, Θ is continuously differentiable as stated. \square

Proposition 2.19. *Assume that $p, q \in [1, 2[$ with $q \leq p$. For every $x \in \mathbb{R}^d$ and $V, \tilde{V} \in \text{Lip}^{\gamma+1}(\mathbb{R}^d; \mathbb{R}^{d+1})$, maps*

$$\begin{aligned}
\varphi : \begin{cases} C^{p\text{-var}}([0, T]; \mathbb{R}^{d+1}) &\longrightarrow C^{p\text{-var}}([0, T]; \mathbb{R}^d) \\ w &\longmapsto \partial_V \pi_V(0, x; w).\tilde{V} \end{cases}, \\
\psi : \begin{cases} C^{p\text{-var}}([0, T]; \mathbb{R}^{d+1}) &\longrightarrow C^{p\text{-var}}([0, T]; \mathbb{R}^{d^2}) \\ w &\longmapsto J_{\leftarrow 0}^w \end{cases} \quad \text{and} \\
\zeta : \begin{cases} C^{p\text{-var}}([0, T]; \mathbb{R}^{d+1}) &\longrightarrow C^{p\text{-var}}([0, T]; \mathbb{R}^{d^2}) \\ w &\longmapsto J_{0\leftarrow}^w \end{cases}
\end{aligned}$$

are continuously differentiable.

Moreover, for every $w \in C^{p\text{-var}}([0, T]; \mathbb{R}^{d+1})$ and $h \in C^{q\text{-var}}([0, T]; \mathbb{R}^{d+1})$, there exists three constants $C_\varphi > 0$, $C_\psi > 0$ and $C_\zeta > 0$, not depending on w and h , such that :

$$\begin{aligned}
\|\partial_w \varphi(w).h\|_{p\text{-var};T} &\leq C_\varphi e^{C_\varphi (\|h\|_{q\text{-var};T}^p + \|w\|_{p\text{-var};T}^p)}, \\
\|\partial_w \psi(w).h\|_{p\text{-var};T} &\leq C_\psi e^{C_\psi (\|h\|_{q\text{-var};T}^p + \|w\|_{p\text{-var};T}^p)} \quad \text{and} \\
\|\partial_w \zeta(w).h\|_{p\text{-var};T} &\leq C_\zeta e^{C_\zeta (\|h\|_{q\text{-var};T}^p + \|w\|_{p\text{-var};T}^p)}.
\end{aligned}$$

Proof. As seen at Proposition 2.16 (step 1 of the proof), for every continuous function $w : [0, T] \rightarrow \mathbb{R}^{d+1}$ of finite p -variation, $\varphi(w)$ satisfies :

$$\varphi(w) = (I^1 \circ I^2 \circ I^3)(w)$$

where, with notations of Proposition 2.16 :

$$I^1 = \pi_{\bar{A}}^1(0, 0; \cdot), \quad I^2 = \int_0^\cdot F_{V, \tilde{V}}(\cdot) d. \quad \text{and} \quad I^3 = \pi_{F_V}[0, (x, 0); \cdot].$$

When $p \in [1, 2[$, for collections of γ -Lipschitz vector fields, the Itô map is continuously differentiable with respect to the driving signal for p -variation topologies (cf. [12], Corollary 11.7).

Then, since F_V and $F_{V,\tilde{V}}$ are collections of γ -Lipschitz vector fields by construction :

$$\begin{aligned} I^2 : C^{p\text{-var}}([0, T]; \mathbb{R}^d) &\longrightarrow C^{p\text{-var}}([0, T]; \mathcal{L}(\mathbb{R}^d) \times \mathbb{R}^d) \text{ and} \\ I^3 : C^{p\text{-var}}([0, T]; \mathbb{R}^{d+1}) &\longrightarrow C^{p\text{-var}}([0, T]; \mathbb{R}^d) \end{aligned}$$

are continuously differentiable. Moreover, by Proposition 2.18 :

$$I^1 : C^{p\text{-var}}([0, T]; \mathcal{L}(\mathbb{R}^d) \times \mathbb{R}^d) \longrightarrow C^{p\text{-var}}([0, T]; \mathbb{R}^d)$$

is continuously differentiable. Therefore, by composition, φ is also continuously differentiable for p -variation topologies.

Consider $w \in C^{p\text{-var}}([0, T]; \mathbb{R}^{d+1})$ and $h \in C^{q\text{-var}}([0, T]; \mathbb{R}^{d+1})$:

$$\partial_w \varphi(w).h = \langle DI^1[(I^2 \circ I^3)(w)]; \langle DI^2[I^3(w)], \partial_w I^3(w).h \rangle \rangle.$$

Then,

$$(18) \quad \begin{aligned} \|\partial_w \varphi(w).h\|_{p\text{-var};T} &\leq \|DI^1[(I^2 \circ I^3)(w)]\|_{\mathcal{L},p} \times \\ &\quad \|DI^2[I^3(w)]\|_{\mathcal{L},p} \times \\ &\quad \|\partial_w I^3(w).h\|_{p\text{-var};T} \end{aligned}$$

where, for every linear maps Λ between p -variation spaces,

$$\|\Lambda\|_{\mathcal{L},p} = \sup_{\|\eta\|_{p\text{-var};T} \leq 1} \|\Lambda.\eta\|_{p\text{-var};T}.$$

Now, let's find an upper bound for each terms of the product on the right hand side of inequality (18) :

- (1) Since $I^3(w) = \pi_{F_V}[0, (x, 0); w]$, by applying Proposition 2.17 to the driving signal w perturbed in the direction h , the collection of γ -Lipschitz vector fields F_V and the control $\omega = \|w\|_{p\text{-var}}^p$:

$$\begin{aligned} \|\partial_w I^3(w).h\|_{\infty;T} &\leq C_\vartheta \times \\ &\quad \exp \left[C_\vartheta \left[\|h\|_{q\text{-var};T}^p + \sup_{\substack{D = \{r_k\} \in D_{0,T} \\ \omega(r_k, r_{k+1}) \leq 1}} \sum_{k=1}^{|D|-1} \omega(r_k, r_{k+1}) \right] \right] \\ &\leq C_\vartheta e^{C_\vartheta (\|h\|_{q\text{-var};T}^p + \|w\|_{p\text{-var};T}^p)} \end{aligned}$$

by super-additivity of ω .

Moreover, by Proposition 2.12, for every $(s, t) \in \Delta_T$ such that $\tilde{\omega}_0(s, t) \leq 1$:

$$\begin{aligned} \|\partial_w I_{s,t}^3(w).h\| &\leq \tilde{C}_{\text{LR}} [1 + \|\partial_w I_{0,s}^3(w).h\|] \tilde{\omega}_0^{1/p}(s, t) e^{\tilde{C}_{\text{LR}} \tilde{\omega}_0(s, t)} \\ &\leq \tilde{C}_{\text{LR}} [1 + \|\partial_w I^3(w).h\|_{\infty;T}] \omega_0^{1/p}(s, t) e^{\tilde{C}_{\text{LR}} \omega_0(0, T)} \end{aligned}$$

where, $\tilde{\omega}_0$ and ω_0 are controls introduced at Proposition 2.17 (cf. Proof).

Therefore,

$$(19) \quad \|\partial_w I^3(w).h\|_{p\text{-var};T} \leq \tilde{C}_\vartheta e^{\tilde{C}_\vartheta (\|h\|_{q\text{-var};T}^p + \|w\|_{p\text{-var};T}^p)}$$

where, $\tilde{C}_\vartheta > 0$ is a constant not depending on w and h .

- (2) Consider $\eta \in C^{p\text{-var}}([0, T]; \mathbb{R}^d)$ such that $\|\eta\|_{p\text{-var}; T} \leq 1$, $\tilde{w} = I^3(w)$ and the control $\tilde{\omega} : \bar{\Delta}_T \rightarrow \mathbb{R}_+$ such that for every $(s, t) \in \Delta_T$,

$$\begin{aligned} \tilde{\omega}^{1/p}(s, t) &= (\alpha \|w\|_{p\text{-var}; s, t}) \vee (\alpha \|w\|_{p\text{-var}; s, t})^p \\ &\geq \|I^3(w)\|_{p\text{-var}; s, t} \end{aligned}$$

by Proposition 2.11 ($\alpha \geq 1$, not depending on w and η).

As explained by P. Friz and N. Victoir at [12], Section 10.6, there exists a collection of vector fields $\Phi_{V, \tilde{V}}$, γ -Lipschitz as $F_{V, \tilde{V}}$, such that :

$$I^2 [I^3(w)] = \pi_{\Phi_{V, \tilde{V}}} [0, 0; I^3(w)].$$

Then, by applying Proposition 2.17 to the driving signal $I^3(w)$ perturbed in the direction η , the collection of γ -Lipschitz vector fields $\Phi_{V, \tilde{V}}$ and the control $\tilde{\omega}$ defined above :

$$\begin{aligned} \|\partial_{\tilde{w}} I^2(\tilde{w}) \cdot \eta\|_{\infty; T} &\leq C_\vartheta \times \\ &\exp \left[C_\vartheta \left[\|\eta\|_{p\text{-var}; T}^p + \sup_{\substack{D = \{r_k\} \in D_{0, T} \\ \tilde{\omega}(r_k, r_{k+1}) \leq 1}} \sum_{k=1}^{|D|-1} \tilde{\omega}(r_k, r_{k+1}) \right] \right] \\ &\leq C_\vartheta \times \\ &\exp \left[C_\vartheta \left[\|\eta\|_{p\text{-var}; T}^p + \sup_{\substack{D = \{r_k\} \in D_{0, T} \\ \omega(r_k, r_{k+1}) \leq 1}} \sum_{k=1}^{|D|-1} \alpha^p \omega(r_k, r_{k+1}) \right] \right] \\ &\leq C_\vartheta e^{C_\vartheta (\|\eta\|_{p\text{-var}; T}^p + \alpha^p \|w\|_{p\text{-var}; T}^p)} \end{aligned}$$

because,

$$\omega \leq \alpha^p \omega \equiv \tilde{\omega} \text{ when } \tilde{\omega} \leq 1.$$

With notations of Proposition 2.17 (cf. Proof) :

$$\begin{aligned} \omega_0 &= \alpha_0 [\|\eta\|_{p\text{-var}; s, t}^p + \tilde{\omega}(s, t)] \\ &= \alpha_0 [\|\eta\|_{p\text{-var}; s, t}^p + (\alpha \|w\|_{p\text{-var}; s, t})^p \vee (\alpha \|w\|_{p\text{-var}; s, t})^{p^2}]. \end{aligned}$$

Then, when $\tilde{\omega}_0 \leq 1$,

$$\tilde{\omega}_0 \equiv \omega_0 \equiv \alpha_0 (\|\eta\|_{p\text{-var}}^p + \alpha^p \omega)$$

and, as at point 1 :

$$\|\partial_{\tilde{w}} I^2(\tilde{w}) \cdot \eta\|_{p\text{-var}; T} \leq \bar{C}_\vartheta e^{\bar{C}_\vartheta (\|\eta\|_{p\text{-var}; T}^p + \|w\|_{p\text{-var}; T}^p)}$$

where, $\bar{C}_\vartheta > 0$ is a constant not depending on w and η .

Therefore,

$$(20) \quad \|\partial_{\tilde{w}} I^2(\tilde{w})\|_{\mathcal{L}, p} \leq \bar{C}_\vartheta e^{\bar{C}_\vartheta (1 + \|w\|_{p\text{-var}; T}^p)}.$$

- (3) Consider $\tilde{\eta} = (H, \eta)$ belonging to $C^{p\text{-var}}([0, T]; \mathcal{L}(\mathbb{R}^d) \times \mathbb{R}^d)$ and satisfying $\|\tilde{\eta}\|_{p\text{-var}; T} \leq 1$, $\tilde{w} = (I^2 \circ I^3)(w)$ and the control $\bar{\omega} : \bar{\Delta}_T \rightarrow \mathbb{R}_+$ such that for every $(s, t) \in \Delta_T$,

$$\begin{aligned} \bar{\omega}^{1/p}(s, t) &= (\tilde{\alpha} \|w\|_{p\text{-var}; s, t}) \vee (\tilde{\alpha} \|w\|_{p\text{-var}; s, t})^p \vee (\tilde{\alpha} \|w\|_{p\text{-var}; s, t})^{p^2} \\ &\geq \|(I^2 \circ I^3)(w)\|_{p\text{-var}; s, t} \end{aligned}$$

by propositions 2.11 and 2.13.

Since $I^1(\tilde{w}) = \pi_A(0, 0; \tilde{w})$, by applying Lemma 2.18 to the driving signal $(I^2 \circ I^3)(w)$ perturbed in the direction $\tilde{\eta}$, the collection of linear vector fields A and the control $\tilde{\omega}$ defined above :

$$\begin{aligned} \|\partial_{\tilde{w}} I^1(\tilde{w}).\tilde{\eta}\|_{\infty;T} &\leq C_\Theta \times \\ &\exp \left[C_\Theta \left[\|\tilde{\eta}\|_{p\text{-var};T}^p + \sup_{\substack{D = \{r_k\} \in D_{0,T} \\ \tilde{\omega}(r_k, r_{k+1}) \leq 1}} \sum_{k=1}^{|D|-1} \tilde{\omega}(r_k, r_{k+1}) \right] \right] \\ &\leq C_\Theta \times \\ &\exp \left[C_\Theta \left[\|\tilde{\eta}\|_{p\text{-var};T}^p + \sup_{\substack{D = \{r_k\} \in D_{0,T} \\ \omega(r_k, r_{k+1}) \leq 1}} \sum_{k=1}^{|D|-1} \alpha^p \omega(r_k, r_{k+1}) \right] \right] \\ &\leq C_\Theta e^{C_\Theta (\|\tilde{\eta}\|_{p\text{-var};T}^p + \alpha^p \|w\|_{p\text{-var};T}^p)} \end{aligned}$$

because,

$$\omega \leq \alpha^p \omega \equiv \tilde{\omega} \text{ when } \tilde{\omega} \leq 1.$$

As at point 2 :

$$\|\partial_{\tilde{w}} I^1(\tilde{w}).\tilde{\eta}\|_{p\text{-var};T} \leq \tilde{C}_\Theta e^{\tilde{C}_\Theta (\|\tilde{\eta}\|_{p\text{-var};T}^p + \|w\|_{p\text{-var};T}^p)}$$

where, $\tilde{C}_\Theta > 0$ is a constant not depending on w and $\tilde{\eta}$.

Therefore,

$$(21) \quad \|\partial_{\tilde{w}} I^1(\tilde{w})\|_{\mathcal{L},p} \leq \tilde{C}_\Theta e^{\tilde{C}_\Theta (1 + \|w\|_{p\text{-var};T}^p)}.$$

In conclusion, via (18), (19), (20) and (21) :

$$\|\partial_w \varphi(w).h\|_{p\text{-var};T} \leq C_\varphi e^{C_\varphi (\|h\|_{q\text{-var};T}^p + \|w\|_{p\text{-var};T}^p)}.$$

The upper bounds for $\|\partial_w \psi(w).h\|_{p\text{-var};T}$ and $\|\partial_w \zeta(w).h\|_{p\text{-var};T}$ are obtained by following the exact same method. \square

3. MALLIAVIN CALCULUS AND GAUSSIAN ROUGH PATHS

As usual (for example in E. Fournié et al. [10] or E. Gobet and R. Müños [15]), in order to compute Greeks without differentiability assumption(s) on F , we need a basic introduction to Malliavin calculus first (cf. D. Nualart [28]). In a second part, we state some results on Gaussian rough paths (cf. [12], Chapter 15 and [11]) and on the integrability of linear RDEs driven by Gaussian signals (cf. P. Friz and S. Riedel [13] and T. Cass, C. Litterer and T. Lyons [2]). We also extend [12], Proposition 20.5 for equations with a drift term.

We work on the probability space $(\Omega, \mathcal{A}, \mathbb{P})$ where $\Omega = C^0([0, T]; \mathbb{R}^d)$, \mathcal{A} is the σ -algebra generated by cylinder sets and \mathbb{P} is the probability measure induced by W on (Ω, \mathcal{A}) .

3.1. Malliavin calculus. On one hand, for $i = 1, \dots, d$, the Cameron-Martin's space of W^i is given by :

$$\mathcal{H}_{W^i}^1 = \{h \in C^0([0, T]; \mathbb{R}) : \exists Z \in \mathcal{A}_{W^i} \text{ s.t. } \forall t \in [0, T], h_t = \mathbb{E}(W_t^i Z)\}$$

with

$$\mathcal{A}_{W^i} = \overline{\text{span} \{W_t^i; t \in [0, T]\}}^{L^2}.$$

More generally,

$$\mathcal{H}_W^1 = \bigoplus_{i=1}^d \mathcal{H}_{W^i}^1$$

is the Cameron-Martin's space of W .

For $i = 1, \dots, d$, let $\langle \cdot, \cdot \rangle_{\mathcal{H}_{W^i}^1}$ be the map defined on $\mathcal{H}_{W^i}^1 \times \mathcal{H}_{W^i}^1$ by :

$$\langle h, \tilde{h} \rangle_{\mathcal{H}_{W^i}^1} = \mathbb{E}(Z \tilde{Z})$$

where,

$$\forall t \in [0, T], h_t = \mathbb{E}(W_t^i Z) \text{ and } \tilde{h}_t = \mathbb{E}(W_t^i \tilde{Z})$$

with $Z, \tilde{Z} \in \mathcal{A}_{W^i}$.

The natural extension of these scalar products on \mathcal{H}_W^1 is denoted by $\langle \cdot, \cdot \rangle_{\mathcal{H}_W^1}$. Equipped with it, \mathcal{H}_W^1 is a Hilbert space.

On the other hand, for $i = 1, \dots, d$, consider the Hilbert space $\mathcal{H}_{W^i} = \overline{\mathcal{E}}^{\langle \cdot, \cdot \rangle_{\mathcal{H}_{W^i}}}$ where \mathcal{E} is the space of \mathbb{R} -valued step functions on $[0, T]$ and $\langle \cdot, \cdot \rangle_{\mathcal{H}_{W^i}}$ is the scalar product defined by :

$$\forall s, t \in [0, T], \langle \mathbf{1}_{[0,s]}, \mathbf{1}_{[0,t]} \rangle_{\mathcal{H}_{W^i}} = \mathbb{E}(W_s^i W_t^i).$$

The natural extension of these scalar products on $\mathcal{H}_W = \mathcal{H}_{W^1} \oplus \dots \oplus \mathcal{H}_{W^d}$ is denoted by $\langle \cdot, \cdot \rangle_{\mathcal{H}_W}$. Equipped with it, \mathcal{H}_W is a Hilbert space too.

For $i = 1, \dots, d$, there exists an isonormal Gaussian process \mathbf{W}^i on \mathcal{H}_{W^i} such that :

$$\forall t \in [0, T], \mathbf{W}^i(\mathbf{1}_{[0,t]}) = W_t^i.$$

Then, we define an isonormal Gaussian process \mathbf{W} on \mathcal{H}_W by :

$$\forall \varphi = (\varphi^1, \dots, \varphi^d) \in \mathcal{H}_W, \mathbf{W}(\varphi) = \sum_{i=1}^d \mathbf{W}^i(\varphi^i).$$

This construction implies that $I = (I^1, \dots, I^d)$ is an isometry between \mathcal{H}_W and \mathcal{H}_W^1 where, for $i = 1, \dots, d$,

$$(22) \quad I^i : \begin{cases} \mathcal{H}_W \longrightarrow \mathcal{H}_{W^i}^1 \\ \varphi = (\varphi^1, \dots, \varphi^d) \longmapsto h = \mathbb{E}[\mathbf{W}^i(\varphi^i) W^i] \end{cases}.$$

Example. Suppose that W is a 1-dimensional Brownian motion. For every $s, t \in [0, T]$,

$$\begin{aligned} I(\mathbf{1}_{[0,t]})(s) &= \mathbb{E}[\mathbf{W}(\mathbf{1}_{[0,t]}) W_s] \\ &= \mathbb{E}(W_t W_s) \\ &= s \wedge t \\ &= \int_0^s \mathbf{1}_{[0,t]}(u) du. \end{aligned}$$

Since $\overline{\mathcal{E}}^{\langle \cdot, \cdot \rangle_{\mathcal{H}_W}} = \mathcal{H}_W$ and isometries I and

$$\varphi \longmapsto \int_0^\cdot \varphi(u) du$$

are continuous on $\mathcal{H}_W = L^2([0, T])$, the previous equality is true on \mathcal{H}_W .

Now, let's remind some basic definitions of Malliavin calculus stated at sections 1.2, 1.3 and 4.1 of [28] :

Let's denote by $C_{\mathbf{p}}^\infty(\mathbb{R}^n; \mathbb{R})$ the space of functions belonging to $C^\infty(\mathbb{R}^n; \mathbb{R})$, with at most polynomial growth and derivatives with at most polynomial growth too ($n \in \mathbb{N}^*$).

Definition 3.1. *The Malliavin derivative of a smooth functional*

$$F = f[\mathbf{W}(h_1), \dots, \mathbf{W}(h_n)],$$

where $n \in \mathbb{N}^*$, $f \in C_{\mathbf{p}}^\infty(\mathbb{R}^n; \mathbb{R})$ and $h_1, \dots, h_n \in \mathcal{H}_W$, is given by :

$$\mathbf{D}F = \sum_{k=1}^n \partial_k f[\mathbf{W}(h_1), \dots, \mathbf{W}(h_n)] h_k.$$

Malliavin's derivative is a closable operator and the domain of its closure in $L^2(\Omega)$ is denoted by $\mathbb{D}^{1,2}$ (cf. [28], Proposition 1.2.1). In the sequel, we also need the two following spaces associated with $\mathbb{D}^{1,2}$:

- The set $\mathbb{D}_{\text{loc}}^{1,2}$ of random variables F such that there exists a sequence $\{(\Omega_n, F_n); n \in \mathbb{N}^*\} \subset \mathcal{A} \times \mathbb{D}^{1,2}$ satisfying almost surely : $\Omega_n \uparrow \Omega$ and $F = F_n$ on Ω_n for every $n \in \mathbb{N}^*$.
- The set $\mathbb{D}^{1,2}(\mathcal{H}_W)$ of stochastic processes u defined on $[0, T]$, such that :

$$\|u\|_{1,2;\mathcal{H}_W}^2 = \mathbb{E}(\|u\|_{\mathcal{H}_W}^2) + \mathbb{E}(\|\mathbf{D}u\|_{\mathcal{H}_W^{\otimes 2}}^2) < \infty.$$

Definition 3.2. *The divergence operator δ is the adjoint of \mathbf{D} :*

- (1) *The domain of δ , denoted by $\text{dom}(\delta)$, is the set of \mathcal{H}_W -valued square integrable random variables $u \in L^2(\Omega; \mathcal{H}_W)$ such that :*

$$\forall F \in \mathbb{D}^{1,2}, |\mathbb{E}(\langle \mathbf{D}F, u \rangle_{\mathcal{H}_W})| \leq M_{DIV} \|F\|_2$$

where, $M_{DIV} > 0$ is a deterministic constant depending only on u .

- (2) *For every $u \in \text{dom}(\delta)$, $\delta(u)$ is the random variable of $L^2(\Omega)$ such that :*

$$\forall F \in \mathbb{D}^{1,2}, \mathbb{E}(\langle \mathbf{D}F, u \rangle_{\mathcal{H}_W}) = \mathbb{E}[F\delta(u)].$$

Note that $\mathbb{D}^{1,2}(\mathcal{H}_W) \subset \text{dom}(\delta)$ (cf. [28], Proposition 1.3.1).

Definition 3.3. *A functional $\varphi : \Omega \rightarrow \mathbb{R}^d$ is \mathcal{H}_W^1 -differentiable if and only if, for almost every $\omega \in \Omega$,*

$$h \in \mathcal{H}_W^1 \mapsto \varphi^i(\omega + h)$$

is continuously differentiable (in the sense of Fréchet) for $i = 1, \dots, d$.

In particular, if φ is \mathcal{H}_W^1 -differentiable, φ belongs to $\mathbb{D}_{\text{loc}}^{1,2}$ (cf. [28], Proposition 4.1.3 and [12], Appendix D.5). Moreover, if $\mathbb{E}(\|\varphi\|^2) < \infty$ and $\mathbb{E}(\|\mathbf{D}\varphi\|_{\mathcal{H}_W}^2) < \infty$, φ belongs to $\mathbb{D}^{1,2}$ (cf. [28], Lemma 4.1.2).

3.2. Gaussian rough paths. On one hand, we remind what conditions the covariance function of W has to satisfy to ensure the existence of a geometric rough path over W . On the other hand, we summarize and extend a little bit probabilistic conclusions of the recent paper of T. Cass, C. Litterer and T. Lyons [2].

Definition 3.4. *A function φ from $[0, T]^2$ into \mathbb{R}^d has finite ρ -variation in 2D sense ($\rho \geq 1$) if and only if,*

$$\sup_{\substack{D_1 = \{r_k^1\} \in D_{0,T} \\ D_2 = \{r_l^2\} \in D_{0,T}}} \sum_{k=1}^{|D_1|-1} \sum_{l=1}^{|D_2|-1} \left\| \varphi \begin{pmatrix} r_k^1 & r_l^2 \\ r_{k+1}^1 & r_{l+1}^2 \end{pmatrix} \right\|^\rho < \infty$$

where

$$\forall t > s, \forall v > u, \varphi \begin{pmatrix} s & u \\ t & v \end{pmatrix} = \varphi(s, u) + \varphi(t, v) - \varphi(s, v) - \varphi(t, u).$$

In the sequel, we assume that W satisfies :

Assumption 3.5. *W is a d -dimensional continuous and centered Gaussian process on $[0, T]$ with independent components, and for $i = 1, \dots, d$, the covariance function of W^i has finite ρ -variation in 2D sense for $\rho \in [1, 2[$.*

Under Assumption 3.5, from [12], Theorem 15.33, a geometric p -rough path \mathbb{W} exists over W for $p \in]2\rho, 4[$.

In order to show Lemma 3.9 and state probabilistic results of [2], the Cameron-Martin's space of W has to satisfy the following assumption :

Assumption 3.6. *There exists $q \geq 1$ such that :*

$$\frac{1}{p} + \frac{1}{q} > 1 \text{ and } \mathcal{H}_W^1 \hookrightarrow C^{q\text{-var}}([0, T]; \mathbb{R}^d).$$

Examples. By [12], Section 20.1, Assumption 3.6 is satisfied if the covariance of W has finite 2D ρ -variation for some $\rho < 3/2$. The fractional Brownian motion with Hurst parameter $H > 1/3$ satisfies this condition. In that particular case, some regularity arguments ensure that it is still true for $H > 1/4$ (cf. [12], question (iii) of Exercice 20.2).

Now, let's talk about new results provided in [2].

The following proposition is a consequence of [2], Theorem 6.4 (and Remark 6.5) used by the authors to prove [2], theorems 6.6 and 6.7 :

Proposition 3.7. *Under assumptions 3.5 and 3.6, for every deterministic constants $C, \alpha, r > 0$,*

$$Ce^{CN_{\alpha, I, p}(\mathbb{W})} \in L^r(\Omega, \mathbb{P}).$$

Corollary 3.8. *Consider $q \in [1, p]$ with $1/p + 1/q > 1$, $h \in C^{q\text{-var}}([0, T]; \mathbb{R}^{d+1})$, $x \in \mathbb{R}^d$ and $V, \tilde{V} \in Lip^\gamma(\mathbb{R}^d; \mathbb{R}^{d+1})$. Under assumptions 3.5 and 3.6,*

$$\|J_{\leftarrow 0}^{\tilde{\mathbb{W}}}\|_{\infty; T}, \|\partial_V \pi_V(0, x; \tilde{\mathbb{W}}) \cdot \tilde{V}\|_{\infty; T} \text{ and } \|D_h \vartheta^{\tilde{\mathbb{W}}}\|_{\infty; T}$$

belong to $L^r(\Omega, \mathbb{P})$ for every $r > 0$.

Proof. On one hand, upper bounds obtained at propositions 2.15 (cf. Remark 3) and 2.16 together with inequality (6) and the previous Proposition 3.7 ensure that :

$$\forall r > 0, \|J_{\leftarrow 0}^{\tilde{\mathbb{W}}}\|_{\infty; T} \in L^r(\Omega, \mathbb{P}) \text{ and } \|\partial_V \pi_V(0, x; \tilde{\mathbb{W}}) \cdot \tilde{V}\|_{\infty; T} \in L^r(\Omega, \mathbb{P}).$$

On the other hand, by Lemma 2.14 together with Proposition 2.17 applied to the driving signal $\tilde{\mathbb{W}}$ perturbed in the direction h , the collection of γ -Lipschitz vector fields V and the control $\omega : \bar{\Delta}_T \rightarrow \mathbb{R}_+$ defined by :

$$\forall (s, t) \in \Delta_T, \omega(s, t) = C(p, 1) [\omega_{\mathbb{W}, p}(s, t) + \|\text{Id}_{[0, T]}\|_{1\text{-var}; s, t}^p],$$

we get :

$$\|D_h \vartheta^{\tilde{\mathbb{W}}}\|_{\infty; T} \leq \tilde{C}_\vartheta e^{\tilde{C}_\vartheta [\|h\|_{q\text{-var}; T}^p + M_{\tilde{C}_\vartheta, I, p}(\mathbb{W})]}$$

where, $\tilde{C}_\vartheta > 0$ is a deterministic constant not depending on \mathbb{W} and h .

Then, inequality (6) together with Proposition 3.7 ensure that :

$$\forall r > 0, \|D_h \vartheta^{\tilde{\mathbb{W}}}\|_{\infty; T} \in L^r(\Omega, \mathbb{P}).$$

□

It is now possible to take (1) in the sense of rough paths. Indeed, formally, equation (1) can be rewritten as follow :

$$dX_t^{x,\sigma} = V(X_t^{x,\sigma}) d\tilde{W}_t$$

where, $X_0^{x,\sigma} = x \in \mathbb{R}^d$ and V is the collection of vector fields on \mathbb{R}^d defined by :

$$\forall y, w \in \mathbb{R}^d, \forall \tau \in \mathbb{R}, V(y).(w, \tau) = b(y)\tau + \sigma(y)w.$$

Since b, σ and their derivatives up to the level $[p]+1$ are bounded under Assumption 1.1, V is a collection of γ -Lipschitz vector fields for $\gamma > p$. From Proposition 2.9 :

$$dX^{x,\sigma} = V(X^{x,\sigma}) d\widetilde{\mathbb{W}},$$

with initial condition x , admits a unique solution $\pi_V(0, x; \widetilde{\mathbb{W}})$.

In that context, we prove the following lemma which extends [12], Proposition 20.5 for $b \neq 0$:

Lemma 3.9. *Under assumptions 1.1, 3.5 and 3.6, for every $x \in \mathbb{R}^d$ and almost every $\omega \in \Omega$,*

$$h \in \mathcal{H}_W^1 \longmapsto X(\omega, h) = \pi_V \left[0, x; \widetilde{\mathbb{W}}(\omega + h) \right]$$

is continuously differentiable in the sense of Fréchet and, in particular :

$$\forall h \in \mathcal{H}_W^1, \forall t \in [0, T], D_h X_t^{x,\sigma} = \int_0^t J_{t \leftarrow s}^{\widetilde{\mathbb{W}}} \sigma(X_s^{x,\sigma}) dh_s$$

with,

$$D_h X_t^{x,\sigma} = \left\{ \frac{d}{d\varepsilon} \pi_V \left[0, x; T_{(\varepsilon h, 0)} \widetilde{\mathbb{W}} \right]_t \right\}_{\varepsilon=0}.$$

Moreover, for every $t \in [0, T]$, $X_t^{x,\sigma}$ belongs to $\mathbb{D}_{loc}^{1,2}$ and

$$\forall h \in \mathcal{H}_W^1, \langle \mathbf{D} X_t^{x,\sigma}, I^{-1}(h) \rangle_{\mathcal{H}_W} = D_h X_t^{x,\sigma}.$$

Proof. On one hand, from P. Friz and N. Victoir [12], Lemma 15.58 (which needs assumptions 3.5 and 3.6), for almost every $\omega \in \Omega$ and every $h \in \mathcal{H}_W^1$,

$$\begin{aligned} \widetilde{\mathbb{W}}(\omega + h) &= S_{[p]} [\mathbb{W}(\omega + h) \oplus \text{Id}_{[0,T]}] \\ &= S_{[p]} [T_h \mathbb{W}(\omega) \oplus \text{Id}_{[0,T]}] \\ &= T_{(h,0)} S_{[p]} [\mathbb{W}(\omega) \oplus \text{Id}_{[0,T]}] \\ &= T_{(h,0)} \widetilde{\mathbb{W}}(\omega). \end{aligned}$$

Then, almost surely :

$$(23) \quad \pi_V \left[0, x; \widetilde{\mathbb{W}}(\cdot + h) \right] = \pi_V \left[0, x; T_{(h,0)} \widetilde{\mathbb{W}} \right].$$

On the other hand, by [12], Theorem 11.6 and Assumption 3.6 :

$$h \in \mathcal{H}_W^1 \longmapsto \pi_V \left[0, x; T_{(h,0)} \widetilde{\mathbb{W}} \right]$$

is continuously differentiable in the sense of Fréchet. Therefore, from equality (23) :

$$h \in \mathcal{H}_W^1 \longmapsto \pi_V \left[0, x; \widetilde{\mathbb{W}}(\cdot + h) \right]$$

is also continuously differentiable in the sense of Fréchet and the two derivatives are matching almost surely.

From the generalized Duhamel's principle (cf. [12], Exercice 11.9) :

$$\begin{aligned} D_h X_t^{x,\sigma} &= \int_0^t J_{t \leftarrow s}^{\tilde{\mathbb{W}}} V(X_s^{x,\sigma}) . d(h_s, 0) \\ &= \int_0^t J_{t \leftarrow s}^{\tilde{\mathbb{W}}} \sigma(X_s^{x,\sigma}) dh_s. \end{aligned}$$

Finally, by Definition 3.3, for every $t \in [0, T]$, $X_t^{x,\sigma}$ is \mathcal{H}_W^1 -differentiable and then belongs to $\mathbb{D}_{\text{loc}}^{1,2}$ by [28], Proposition 4.1.3 or [12], Appendix D.5 :

$$\forall h \in \mathcal{H}_W^1, \langle \mathbf{D}X_t^{x,\sigma}, I^{-1}(h) \rangle_{\mathcal{H}_W} = D_h X_t^{x,\sigma}.$$

□

4. SENSITIVITY WITH RESPECT TO THE INITIAL CONDITION

In this section, b and σ are fixed. Then, put $X^x = X^{x,\sigma}$ and $f_T(x) = f_T(x, \sigma)$ for every $x \in \mathbb{R}^d$.

In order to establish the second part of Theorem 4.3 and Corollary 4.4, σ and \mathcal{H}_W^1 have to satisfy respectively :

Assumption 4.1. *For every $y \in \mathbb{R}^d$, $\sigma(y)$ is a non singular matrix and σ^{-1} is bounded.*

Assumption 4.2. *The Cameron-Martin's space \mathcal{H}_W^1 satisfies :*

$$C_0^1([0, T]; \mathbb{R}^d) \subset \mathcal{H}_W^1.$$

Moreover, there exists $C_{\mathcal{H}_W^1} > 0$ such that :

$$\forall h \in C_0^1([0, T]; \mathbb{R}^d), \|h\|_{\mathcal{H}_W^1} \leq C_{\mathcal{H}_W^1} \|h\|_{\infty; T}.$$

Remarks :

- (1) In the sequel, keep in mind that C_{RDE} and \bar{C}_{IC} are deterministic constants, not depending on the initial condition.
- (2) For example, the fractional Brownian motion satisfies Assumption 4.2 (cf. [12], Remark 15.10).

Theorem 4.3. *Under assumptions 1.1, 1.2, 3.5 and 3.6, f_T is differentiable on \mathbb{R}^d . Moreover, under assumptions 4.1 and 4.2, for every $x, v \in \mathbb{R}^d$, there exists a d -dimensional stochastic process $h^{x,v}$ defined on $[0, T]$ such that :*

$$(24) \quad Df_T(x).v = \mathbb{E}[\langle \mathbf{D}(F \circ X_T^x), I^{-1}(h^{x,v}) \rangle_{\mathcal{H}_W}].$$

Proof. On one hand, under assumptions 1.1, 1.2, 3.5 and 3.6, we show that f_T is differentiable on \mathbb{R}^d and

$$(25) \quad \forall x, v \in \mathbb{R}^d, Df_T(x).v = \mathbb{E}[\langle DF(X_T^x), DX_T^x.v \rangle].$$

On the other hand, by adding assumptions 4.1 and 4.2, we obtain equality (24) via Lemma 3.9 :

- (1) For every $\varepsilon \in]0, 1]$, $\alpha > 0$ and $x, v \in \mathbb{R}^d$,

$$\begin{aligned} \frac{|F(X_T^{x+\varepsilon v}) - F(X_T^x)|}{\varepsilon} &= \left| \int_0^1 \langle DF(X_T^{x+\theta\varepsilon v}), DX_T^{x+\theta\varepsilon v}.v \rangle d\theta \right| \\ &\leq \|v\| \int_0^1 \|DF(X_T^{x+\theta\varepsilon v})\|_{\mathcal{L}} \|DX_T^{x+\theta\varepsilon v}\|_{\infty; T} d\theta \\ &\leq \bar{C}_{\text{IC}} \|v\| e^{\bar{C}_{\text{IC}} M_{\alpha, I, p}(\mathbb{W})} \int_0^1 \|DF(X_T^{x+\theta\varepsilon v})\|_{\mathcal{L}} d\theta \end{aligned}$$

by Proposition 2.15 (cf. Remark 3) and Taylor's formula.

Since F satisfies Assumption 1.2, for every $\theta \in [0, 1]$,

$$\|DF(X_T^{x+\theta\varepsilon v})\|_{\mathcal{L}} \leq C_F (1 + \|X_T^{x+\theta\varepsilon v}\|)^{N_F}.$$

Then, by Proposition 2.9 and the triangle inequality :

$$\begin{aligned} \|DF(X_T^{x+\theta\varepsilon v})\|_{\mathcal{L}} &\leq C_F (1 + \|x + \theta\varepsilon v\| + \|X_T^{x+\theta\varepsilon v}\|_{p\text{-var};T})^{N_F} \\ &\leq C_F [1 + \|x\| + \|v\| + \\ &\quad C_{\text{RDE}}(\|V\|_{\text{lip}^{\gamma-1}} \|\widetilde{\mathbb{W}}\|_{p\text{-var};T} \vee \|V\|_{\text{lip}^{\gamma-1}}^p \|\widetilde{\mathbb{W}}\|_{p\text{-var};T}^p)]^{N_F}. \end{aligned}$$

Since \mathbb{W} is a Gaussian geometric p -rough path satisfying Assumption 3.6, from Proposition 3.7, inequality (6) (cf. [2], Proposition 4.6), Lemma 2.14, the rough paths extension of Fernique's theorem ([12], Theorem 15.33) and the Cauchy-Schwarz inequality :

$$\varepsilon \in]0, 1] \mapsto \frac{|F(X_T^{x+\varepsilon v}) - F(X_T^x)|}{\varepsilon}$$

is bounded by an integrable random variable which does not depend on ε . Therefore, (25) is true by Lebesgue's theorem.

- (2) For every $x, v \in \mathbb{R}^d$, let $h^{x,v}$ be the stochastic process defined on $[0, T]$ by :

$$\forall t \in [0, T], h_t^{x,v} = \frac{1}{T} \int_0^t \sigma^{-1}(X_s^x) J_{s \leftarrow 0}^{\widetilde{\mathbb{W}}} v ds.$$

Then, Assumption 4.2 implies that $h^{x,v}$ is a \mathcal{H}_W^1 -valued random variable and from Lemma 3.9 :

$$\begin{aligned} D_{h^{x,v}} X_T^x &= \int_0^T J_{T \leftarrow s}^{\widetilde{\mathbb{W}}} \sigma(X_s^x) dh_s^{x,v} \\ &= DX_T^x.v. \end{aligned}$$

Therefore, via the chain rule :

$$\begin{aligned} Df_T(x).v &= \mathbb{E}[DF(X_T^x).D_{h^{x,v}} X_T^x] \\ &= \mathbb{E}[D_{h^{x,v}}(F \circ X_T^x)] \\ &= \mathbb{E}[\langle \mathbf{D}(F \circ X_T^x), I^{-1}(h^{x,v}) \rangle_{\mathcal{H}_W}]. \end{aligned}$$

□

Corollary 4.4. *Under assumptions 1.1, 1.2, 4.1 and 4.2 for $p+1$ ($p \in [1, 2[$), if W is of finite p -variation, then $I^{-1}(h^{x,v}) \in \text{dom}(\delta)$ and*

$$(26) \quad Df_T(x).v = \mathbb{E}[F(X_T^x) \delta[I^{-1}(h^{x,v})]].$$

Equality (26) is still true if Assumption 1.2 is replaced by Assumption 1.3.

Proof. Since W is of finite p -variation with $p \in [1, 2[$, its covariance function is of finite p -variation in 2D sense and then, by [12], Theorem 15.7, $\mathcal{H}_W^1 \hookrightarrow C^{p\text{-var}}([0, T]; \mathbb{R}^d)$. Therefore, assumptions 3.5 and 3.6 are satisfied for $\rho = q = p$, and by Theorem 4.3 :

$$Df_T(x).v = \mathbb{E}[\langle \mathbf{D}(F \circ X_T^x), I^{-1}(h^{x,v}) \rangle_{\mathcal{H}_W}]$$

where,

$$h^{x,v} = \frac{1}{T} \int_0^\cdot \mu_s(\tilde{W}) ds, \mu(\tilde{W}) = \sigma^{-1}[\vartheta(\tilde{W})] \psi^v(\tilde{W}) \text{ and } \psi^v = \psi(\cdot).v$$

with notations of propositions 2.17 and 2.19, in the context of equation (1).

On one hand, let show that $I^{-1}(h^{x,v})$ belongs to $\mathbb{D}^{1,2}(\mathcal{H}_W) \subset \text{dom}(\delta)$:

- (1) By propositions 2.17 and 2.19, μ is continuously differentiable. Then,

$$\partial_w \mu(w).(\lambda, 0) = \langle (D\sigma^{-1})[\vartheta(w)], \partial_w \vartheta(w).(\lambda, 0) \rangle \psi^v(w) + \sigma^{-1}[\vartheta(w)] \partial_w \psi^v(w).(\lambda, 0)$$

for every $w \in C^{p\text{-var}}([0, T]; \mathbb{R}^{d+1})$ and every $\lambda \in \mathcal{H}_W^1 \hookrightarrow C^{p\text{-var}}([0, T]; \mathbb{R}^d)$.

For every $\varepsilon \in]0, 1]$ and $t \in [0, T]$, by Taylor's formula and propositions 2.15, 2.17 and 2.19 :

$$\begin{aligned} \frac{\|\mu_t[\tilde{W} + \varepsilon(\lambda, 0)] - \mu_t(\tilde{W})\|}{\varepsilon} &= \left\| \int_0^1 D\mu_t[\tilde{W} + \varepsilon\theta(\lambda, 0)].(\lambda, 0) d\theta \right\| \\ &\leq C \left[\int_0^1 \|D\vartheta[\tilde{W} + \varepsilon\theta(\lambda, 0)].(\lambda, 0)\|_{\infty; T} \times \right. \\ &\quad \left. \|\psi^v[\tilde{W} + \varepsilon\theta(\lambda, 0)]\|_{\infty; T} d\theta + \int_0^1 \|D\psi^v[\tilde{W} + \varepsilon\theta(\lambda, 0)].(\lambda, 0)\|_{\infty; T} d\theta \right] \\ (27) \quad &\leq \tilde{C} \exp \left[\tilde{C} \left(\|\lambda\|_{\mathcal{H}_W^1}^p + \|W\|_{p\text{var}; T}^p \right) \right] \end{aligned}$$

where, $C > 0$ and $\tilde{C} > 0$ are two deterministic constants, not depending on ε , t , λ and W .

Therefore, by Lebesgue's theorem and [12], Lemma 15.58, $h^{x,v}$ is \mathcal{H}_W^1 -differentiable as a \mathcal{H}_W^1 -valued random variable. Moreover, with notations of Lemma 3.9 :

$$\forall \lambda \in \mathcal{H}_W^1, D_\lambda h^{x,v} = \frac{1}{T} \int_0^\cdot D_\lambda \mu_s ds.$$

In conclusion, $I^{-1}(h^{x,v}) \in \mathbb{D}_{\text{loc}}^{1,2}(\mathcal{H}_W)$.

- (2) By using successively that $I : \mathcal{H}_W \rightarrow \mathcal{H}_W^1$ is an isometry, Assumption 4.2 and Assumption 4.1 :

$$\begin{aligned} \mathbb{E} [\|I^{-1}(h^{x,v})\|_{\mathcal{H}_W}^2] &= \mathbb{E} (\|h^{x,v}\|_{\mathcal{H}_W^1}^2) \\ &\leq C_{\mathcal{H}_W^1} \mathbb{E} \left(\sup_{t \in [0, T]} \|\dot{h}_t^{x,v}\|^2 \right) \\ &\leq \bar{C} \mathbb{E} \left(\sup_{t \in [0, T]} \|J_{t \leftarrow 0}^{\tilde{W}}\|_{\mathcal{M}}^2 \right) \end{aligned}$$

where, $\bar{C} > 0$ is a deterministic constant depending on v and σ .

Then, by Corollary 3.8 :

$$\mathbb{E} [\|I^{-1}(h^{x,v})\|_{\mathcal{H}_W}^2] < \infty.$$

- (3) By using successively that $I : \mathcal{H}_W \rightarrow \mathcal{H}_W^1$ is an isometry, Assumption 4.2 and inequality (27) :

$$\begin{aligned}
\mathbb{E} \left[\|\mathbf{D}[I^{-1}(h^{x,v})]\|_{\mathcal{H}_W^{\otimes 2}}^2 \right] &\leq \mathbb{E} \left[\left(\sup_{\|\lambda^1 \otimes \lambda^2\|_{\mathcal{H}_W^{\otimes 2}} \leq 1} |\langle D_{I(\lambda^1)}[I^{-1}(h^{x,v})], \lambda^2 \rangle_{\mathcal{H}_W}| \right)^2 \right] \\
&= \mathbb{E} \left[\left(\sup_{\|\lambda^1 \otimes \lambda^2\|_{\mathcal{H}_W^{\otimes 2}} \leq 1} |\langle I^{-1}[D_{I(\lambda^1)}(h^{x,v})], \lambda^2 \rangle_{\mathcal{H}_W}| \right)^2 \right] \\
&= \mathbb{E} \left[\left(\sup_{\|\lambda^1 \otimes \lambda^2\|_{\mathcal{H}_W^{\otimes 2}} \leq 1} |\langle D_{I(\lambda^1)}(h^{x,v}), I(\lambda^2) \rangle_{\mathcal{H}_W^1}| \right)^2 \right] \\
&\leq \mathbb{E} \left(\sup_{\|\lambda\|_{\mathcal{H}_W^1} \leq 1} \|D_\lambda h^{x,v}\|_{\mathcal{H}_W^1}^2 \right) \\
&\leq \frac{C_{\mathcal{H}_W^1}}{T} \mathbb{E} \left(\sup_{\|\lambda\|_{\mathcal{H}_W^1} \leq 1} \sup_{t \in [0, T]} \|D_\lambda \mu_t\|^2 \right) \\
&\leq \frac{C_{\mathcal{H}_W^1}}{T} \tilde{C} \mathbb{E} \left[\exp \left[\tilde{C} \left(1 + \|W\|_{p\text{var}; T}^p \right) \right] \right].
\end{aligned}$$

Then, by Fernique's theorem (cf. [12], Theorem 15.33) :

$$\mathbb{E} \left[\|\mathbf{D}[I^{-1}(h^{x,v})]\|_{\mathcal{H}_W^{\otimes 2}}^2 \right] < \infty.$$

On the other hand, since $C_K^\infty(\mathbb{R}^d; \mathbb{R})$ is dense in $L^2(\mathbb{R}^d)$ and equality (26) does not involve DF , by walking the exact same way that E. Fournié et al. at the proof of [10], Proposition 3.2 (ii), one can show that (26) is still true under Assumption 1.3. Indeed, functions of $C_K^\infty(\mathbb{R}^d; \mathbb{R})$ are bounded with bounded derivatives and then satisfy Assumption 1.2 in particular. \square

5. SENSITIVITY WITH RESPECT TO THE VOLATILITY FUNCTION

In this section, $x \in \mathbb{R}^d$ is fixed. Then, put $V_{b,\sigma} = V$, $X^\sigma = X^{x,\sigma}$ and $f_T(\sigma) = f_T(x, \sigma)$ for every $\sigma \in \Sigma$ (characteristics of Σ are implicitly specified in each results of this section).

For every $\tilde{\sigma} \in \Sigma$, consider $V_{\tilde{\sigma}}$ the collection of vector fields on \mathbb{R}^d defined by :

$$\forall y, w \in \mathbb{R}^d, \forall a \in \mathbb{R}, V_{\tilde{\sigma}}(y) \cdot (w, a) = \tilde{\sigma}(y)w.$$

By Proposition 2.16 and Corollary 3.8, $\sigma \in \Sigma \mapsto X^\sigma$ is continuously differentiable under Assumption 1.1 with

$$DX^\sigma \cdot \tilde{\sigma} = \partial_V \pi_{V_{b,\sigma}}(0, x; \widetilde{\mathbb{W}}) \cdot V_{\tilde{\sigma}}$$

and, under assumptions 3.5 and 3.6, $\|DX^\sigma \cdot \tilde{\sigma}\|_{\infty; T}$ admits an L^r -upper bound for every $r > 0$:

$$\|DX^\sigma \cdot \tilde{\sigma}\|_{\infty; T} \leq C_{\text{VF}}(\sigma, \tilde{\sigma}) e^{C_{\text{VF}}(\sigma, \tilde{\sigma}) M_{\alpha_{\text{VF}}(\sigma, \tilde{\sigma}), I, p}(\mathbb{W})}$$

where, $C_{\text{VF}}(\sigma, \tilde{\sigma}) = C_{\text{VF}}(V_{b,\sigma}, V_{\tilde{\sigma}})$ and $\alpha_{\text{VF}}(\sigma, \tilde{\sigma}) = \alpha_{\text{VF}}(V_{b,\sigma}, V_{\tilde{\sigma}})$.

Remark. Note that $\|F_{V_{b,\sigma}, V_{\tilde{\sigma}}}\|_{\text{lip}^{\gamma-1}} > 0$ and $\|F_{V_{b,\sigma}}\|_{\text{lip}^{\gamma-1}} > 0$ for every functions $\sigma, \tilde{\sigma} \in \Sigma^*$. It follows that :

$$\theta \in [0, 1] \mapsto \|F_{V_{b,\sigma+\theta\tilde{\sigma}}}\|_{\text{lip}^{\gamma-1}} \text{ and } \theta \in [0, 1] \mapsto \|F_{V_{b,\sigma+\theta\tilde{\sigma}}, V_{\tilde{\sigma}}}\|_{\text{lip}^{\gamma-1}}$$

are bounded with bounded inverses. Then, with notations of Proposition 2.16 (cf. Proof), the way $\alpha_0(\sigma, \tilde{\sigma}) = \alpha_0(V_{b,\sigma}, V_{\tilde{\sigma}})$ involves $\|F_{V_{b,\sigma}, V_{\tilde{\sigma}}}\|_{\text{lip}^{\gamma-1}}$ and $\|F_{V_{b,\sigma}}\|_{\text{lip}^{\gamma-1}}$ implies that :

$$\theta \in [0, 1] \longmapsto \alpha_0(\sigma + \theta\tilde{\sigma}, \tilde{\sigma})$$

is a bounded function with bounded inverse.

Therefore,

$$\theta \in [0, 1] \longmapsto \alpha_{\text{VF}}(\sigma + \theta\tilde{\sigma}, \tilde{\sigma}) \text{ and } \theta \in [0, 1] \longmapsto C_{\text{VF}}(\sigma + \theta\tilde{\sigma}, \tilde{\sigma})$$

are deterministic bounded functions.

Theorem 5.1. *Under assumptions 1.1, 1.2, 3.5 and 3.6, f_T is differentiable on Σ . Moreover, for every $\sigma, \tilde{\sigma} \in \Sigma$, under assumptions 4.1 and 4.2, there exists a d -dimensional stochastic process $\eta^{\sigma, \tilde{\sigma}}$ defined on $[0, T]$ such that :*

$$(28) \quad Df_T(\sigma). \tilde{\sigma} = \mathbb{E}[\langle \mathbf{D}(F \circ X_T^\sigma), I^{-1}(\eta^{\sigma, \tilde{\sigma}}) \rangle_{\mathcal{H}_W}].$$

Proof. On one hand, under assumptions 1.1, 1.2, 3.5 and 3.6, we show that f_T is differentiable on Σ and

$$(29) \quad \forall \sigma, \tilde{\sigma} \in \Sigma, Df_T(\sigma). \tilde{\sigma} = \mathbb{E}[\langle DF(X_T^\sigma), DX_T^\sigma. \tilde{\sigma} \rangle].$$

On the other hand, by adding assumptions 4.1 and 4.2, we obtain equality (28) via Lemma 3.9 :

- (1) By Proposition 2.16, Taylor's formula and Assumption 1.2, for every $\varepsilon \in]0, 1]$ and $\sigma, \tilde{\sigma} \in \Sigma$,

$$\begin{aligned} \frac{|F(X_T^{\sigma+\varepsilon\tilde{\sigma}}) - F(X_T^\sigma)|}{\varepsilon} &= \left| \int_0^1 \langle DF(X_T^{\sigma+\theta\varepsilon\tilde{\sigma}}), DX_T^{\sigma+\theta\varepsilon\tilde{\sigma}}. \tilde{\sigma} \rangle d\theta \right| \\ &\leq \int_0^1 \|DF(X_T^{\sigma+\theta\varepsilon\tilde{\sigma}})\|_{\mathcal{L}} \|DX_T^{\sigma+\theta\varepsilon\tilde{\sigma}}. \tilde{\sigma}\| d\theta \\ &\leq C_F \int_0^1 (1 + \|X_T^{\sigma+\theta\varepsilon\tilde{\sigma}}\|)^{N_F} \|DX_T^{\sigma+\theta\varepsilon\tilde{\sigma}}. \tilde{\sigma}\|_{\infty;T} d\theta. \end{aligned}$$

Since $b, \sigma, \tilde{\sigma}$ and their derivatives up to the level $[p] + 1$ are bounded and $\theta, \varepsilon \in [0, 1]$, from the remark above, there exists a deterministic constant $\tilde{C}_{\text{VF}}(\sigma, \tilde{\sigma}) > 0$, not depending on θ and ε , such that :

$$\begin{aligned} &\|V_{b,\sigma+\theta\varepsilon\tilde{\sigma}}\|_{\text{lip}^{\gamma-1}} + \|V_{b,\sigma+\theta\varepsilon\tilde{\sigma}}\|_{\text{lip}^{\gamma-1}}^p + C_{\text{RDE}} + \\ &C_{\text{VF}}(\sigma + \theta\varepsilon\tilde{\sigma}, \tilde{\sigma}) + \alpha_{\text{VF}}(\sigma + \theta\varepsilon\tilde{\sigma}, \tilde{\sigma}) \leq \tilde{C}_{\text{VF}}(\sigma, \tilde{\sigma}). \end{aligned}$$

Then, from propositions 2.9 and 2.16, respectively :

$$\|X^{\sigma+\theta\varepsilon\tilde{\sigma}}\|_{p\text{-var};T} \leq \tilde{C}_{\text{VF}}^2(\sigma, \tilde{\sigma})(\|\widetilde{\mathbb{W}}\|_{p\text{-var};T} \vee \|\widetilde{\mathbb{W}}\|_{p\text{-var};T}^p)$$

and

$$\|DX^{\sigma+\theta\varepsilon\tilde{\sigma}}. \tilde{\sigma}\|_{\infty;T} \leq \tilde{C}_{\text{VF}}(\sigma, \tilde{\sigma}) e^{\tilde{C}_{\text{VF}}(\sigma, \tilde{\sigma}) M_{\tilde{C}_{\text{VF}}(\sigma, \tilde{\sigma}), I, p}(\mathbb{W})}.$$

Since \mathbb{W} is a Gaussian geometric p -rough path satisfying Assumption 3.6, from Proposition 3.7, inequality (6) (cf. [2], Proposition 4.6), Lemma 2.14, the rough paths extension of Fernique's theorem ([12], Theorem 15.33) and the Cauchy-Schwarz inequality :

$$\varepsilon \in]0, 1] \longmapsto \frac{|F(X_T^{\sigma+\varepsilon\tilde{\sigma}}) - F(X_T^\sigma)|}{\varepsilon}$$

is bounded by an integrable random variable which does not depend on ε . Therefore, (29) is true by Lebesgue's theorem.

- (2) For every $\sigma, \tilde{\sigma} \in \Sigma$ such that σ satisfies Assumption 4.1, let $\eta^{\sigma, \tilde{\sigma}}$ be the stochastic process defined on $[0, T]$ by :

$$\forall t \in [0, T], \eta_t^{\sigma, \tilde{\sigma}} = \frac{1}{T} \int_0^t \sigma^{-1}(X_s^\sigma) J_{s \leftarrow T}^{\tilde{W}} DX_T^\sigma \cdot \tilde{\sigma} ds.$$

Then, Assumption 4.2 implies that $\eta^{\sigma, \tilde{\sigma}}$ is a \mathcal{H}_W^1 -valued random variable and from Lemma 3.9 :

$$\begin{aligned} D_{\eta^{\sigma, \tilde{\sigma}}} X_T^\sigma &= \int_0^T J_{T \leftarrow s}^{\tilde{W}} \sigma(X_s^\sigma) d\eta_s^{\sigma, \tilde{\sigma}} \\ &= DX_T^\sigma \cdot \tilde{\sigma}. \end{aligned}$$

Therefore, via the chain rule :

$$\begin{aligned} Df_T(\sigma) \cdot \tilde{\sigma} &= \mathbb{E}[DF(X_T^\sigma) \cdot D_{\eta^{\sigma, \tilde{\sigma}}} X_T^\sigma] \\ &= \mathbb{E}[D_{\eta^{\sigma, \tilde{\sigma}}}(F \circ X_T^\sigma)] \\ &= \mathbb{E}[\langle \mathbf{D}(F \circ X_T^\sigma), I^{-1}(\eta^{\sigma, \tilde{\sigma}}) \rangle_{\mathcal{H}_W}]. \end{aligned}$$

□

Corollary 5.2. *Under assumptions 1.1, 1.2, 4.1 and 4.2 for $p+1$ ($p \in [1, 2[$), if W is of finite p -variation, then $I^{-1}(\eta^{\sigma, \tilde{\sigma}}) \in \text{dom}(\delta)$ and*

$$(30) \quad Df_T(\sigma) \cdot \tilde{\sigma} = \mathbb{E}[F(X_T^\sigma) \delta[I^{-1}(\eta^{\sigma, \tilde{\sigma}})]].$$

Equality (30) is still true if Assumption 1.2 is replaced by Assumption 1.3.

Proof. As at Corollary 4.4, assumptions 3.5 and 3.6 are satisfied for $\rho = q = p$, and by Theorem 5.1 :

$$Df_T(\sigma) \cdot \tilde{\sigma} = \mathbb{E}[\langle \mathbf{D}(F \circ X_T^\sigma), I^{-1}(\eta^{\sigma, \tilde{\sigma}}) \rangle_{\mathcal{H}_W}]$$

where,

$$\eta^{\sigma, \tilde{\sigma}} = \frac{1}{T} \int_0^\cdot \xi_s(\tilde{W}) ds \text{ and } \xi(\tilde{W}) = \sigma^{-1}[\vartheta(\tilde{W})] \psi(\tilde{W}) \zeta_T(\tilde{W}) \varphi_T(\tilde{W})$$

with notations of propositions 2.17 and 2.19, in the context of equation (1).

By propositions 2.17 and 2.19, ξ is continuously differentiable. Then,

$$\begin{aligned} \partial_w \xi(w) \cdot (\lambda, 0) &= \langle (D\sigma^{-1})[\vartheta(w)], \partial_w \vartheta(w) \cdot (\lambda, 0) \rangle \psi(w) \zeta_T(w) \varphi_T(w) + \\ &\quad \sigma^{-1}[\vartheta(w)] [\partial_w \vartheta(w) \cdot (\lambda, 0)] \zeta_T(w) \varphi_T(w) + \\ &\quad \sigma^{-1}[\vartheta(w)] \psi(w) [\partial_w \zeta_T(w) \cdot (\lambda, 0)] \varphi_T(w) + \\ &\quad \sigma^{-1}[\vartheta(w)] \psi(w) \zeta_T(w) \partial_w \varphi_T(w) \cdot (\lambda, 0) \end{aligned}$$

for every $w \in C^{p\text{-var}}([0, T]; \mathbb{R}^{d+1})$ and every $\lambda \in \mathcal{H}_W^1 \hookrightarrow C^{p\text{-var}}([0, T]; \mathbb{R}^d)$.

Therefore, by propositions 2.15, 2.16, 2.17 and 2.19 :

$$\begin{aligned} \|\partial_w \xi(w) \cdot (\lambda, 0)\|_{\infty; T} &\leq C[\|\partial_w \vartheta(w) \cdot (\lambda, 0)\|_{\infty; T} \|\psi(w)\|_{\infty; T} \|\zeta(w)\|_{\infty; T} \|\varphi(w)\|_{\infty; T} + \\ &\quad \|\partial_w \vartheta(w) \cdot (\lambda, 0)\|_{\infty; T} \|\zeta(w)\|_{\infty; T} \|\varphi(w)\|_{\infty; T} + \\ &\quad \|\psi(w)\|_{\infty; T} \|\partial_w \zeta(w) \cdot (\lambda, 0)\|_{\infty; T} \|\varphi(w)\|_{\infty; T} + \\ &\quad \|\psi(w)\|_{\infty; T} \|\zeta(w)\|_{\infty; T} \|\partial_w \varphi(w) \cdot (\lambda, 0)\|_{\infty; T}] \\ (31) \quad &\leq \tilde{C} \exp \left[\tilde{C} \left(\|\lambda\|_{\mathcal{H}_W^1}^p + \|w\|_{p\text{-var}; T}^p \right) \right] \end{aligned}$$

where, $C > 0$ and $\tilde{C} > 0$ are two deterministic constants, not depending on λ and w .

Let show that $I^{-1}(\eta^{\sigma, \tilde{\sigma}})$ belongs to $\mathbb{D}^{1,2}(\mathcal{H}_W) \subset \text{dom}(\delta)$:

- (1) For every $\varepsilon \in]0, 1]$ and $t \in [0, T]$, by Taylor's formula and inequality (31) with $w = \tilde{W} + \varepsilon\theta(\lambda, 0)$ for $\theta \in [0, 1]$:

$$(32) \quad \frac{\|\xi_t[\tilde{W} + \varepsilon(\lambda, 0)] - \xi_t(\tilde{W})\|}{\varepsilon} = \left\| \int_0^1 D\xi_t[\tilde{W} + \varepsilon\theta(\lambda, 0)] \cdot (\lambda, 0) d\theta \right\| \leq \bar{C} \exp \left[\bar{C} \left(\|\lambda\|_{\mathcal{H}_W^1}^p + \|W\|_{p\text{var};T}^p \right) \right]$$

where, $\bar{C} > 0$ is a deterministic constant, not depending on ε , t , λ and W .

Therefore, by Lebesgue's theorem and [12], Lemma 15.58, $\eta^{\sigma, \tilde{\sigma}}$ is \mathcal{H}_W^1 -differentiable as a \mathcal{H}_W^1 -valued random variable. Moreover, with notations of Lemma 3.9 :

$$\forall \lambda \in \mathcal{H}_W^1, D_\lambda \eta^{\sigma, \tilde{\sigma}} = \frac{1}{T} \int_0^T D_\lambda \xi_t dt.$$

In conclusion, $I^{-1}(\eta^{\sigma, \tilde{\sigma}}) \in \mathbb{D}_{\text{loc}}^{1,2}(\mathcal{H}_W)$.

- (2) As at Corollary 4.4, there exists a deterministic constant $\hat{C} > 0$ such that :

$$\mathbb{E} [\|I^{-1}(\eta^{\sigma, \tilde{\sigma}})\|_{\mathcal{H}_W}^2] = \mathbb{E} (\|\eta^{\sigma, \tilde{\sigma}}\|_{\mathcal{H}_W^1}^2) \leq \hat{C} \mathbb{E} \left[\sup_{t \in [0, T]} \|\psi_t(\tilde{W})\|_{\mathcal{M}}^2 \|\zeta_T(\tilde{W})\|_{\mathcal{M}}^2 \|\varphi_T(\tilde{W})\|^2 \right].$$

Then, by propositions 2.15 and 2.16 :

$$\mathbb{E} [\|I^{-1}(\eta^{\sigma, \tilde{\sigma}})\|_{\mathcal{H}_W}^2] < \infty.$$

- (3) Still as at Corollary 4.4, by inequality (32) :

$$\begin{aligned} \mathbb{E} [\|\mathbf{D}[I^{-1}(\eta^{\sigma, \tilde{\sigma}})]\|_{\mathcal{H}_W^{\otimes 2}}^2] &\leq \mathbb{E} \left(\sup_{\|\lambda\|_{\mathcal{H}_W^1} \leq 1} \|D_\lambda \eta^{\sigma, \tilde{\sigma}}\|_{\mathcal{H}_W^1}^2 \right) \\ &\leq \frac{C_{\mathcal{H}_W^1}}{T} \mathbb{E} \left(\sup_{\|\lambda\|_{\mathcal{H}_W^1} \leq 1} \sup_{t \in [0, T]} \|D_\lambda \xi_t\|^2 \right) \\ &\leq \frac{C_{\mathcal{H}_W^1}}{T} \bar{C} \mathbb{E} \left[\exp \left[\bar{C} \left(1 + \|W\|_{p\text{var};T}^p \right) \right] \right]. \end{aligned}$$

Then, by Fernique's theorem (cf. [12], Theorem 15.33) :

$$\mathbb{E} [\|\mathbf{D}[I^{-1}(\eta^{\sigma, \tilde{\sigma}})]\|_{\mathcal{H}_W^{\otimes 2}}^2] < \infty.$$

Equality (30) is still true if Assumption 1.2 is replaced by Assumption 1.3 : same ideas that at Proposition 4.3. \square

6. FRACTIONAL BROWNIAN MOTION

This section presents elementary properties of the fractional Brownian motion and its representation as a Volterra process that has been established by L. Decreusefond and A. Ustunel in [7] (see also D. Nualart [28]). We also deduce an expression of the isometry defined at equation (22) from that last representation.

Definition 6.1. *A fractional Brownian motion with Hurst parameter $H \in]0, 1[$ is a continuous and centered Gaussian process B^H such that :*

$$\forall s, t \in \mathbb{R}_+, \text{cov}(B_t^H, B_s^H) = \frac{1}{2} (|t|^{2H} + |s|^{2H} - |t - s|^{2H}).$$

Remarks :

- (1) B^H is H -self-similar. It means that for every $a > 0$, $(B_{at}, t \in \mathbb{R}_+)$ and $(a^H B_t, t \in \mathbb{R}_+)$ have the same distribution.
- (2) Unfortunately, when $H \neq 1/2$, B^H is not a semimartingale (cf. [28], Proposition 5.1.1).

Now, let's introduce the two fundamental operators of the fractional calculus (cf. S. Samko et al. [35]) :

Definition 6.2. Let ψ be a function from \mathbb{R}_+ into \mathbb{R} . For a given $\alpha \in]0, 1]$, if

$$l^\alpha(\psi)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \psi(s) ds$$

exists for every $t \in \mathbb{R}_+$, $l^\alpha(\psi)$ is the α -fractional integral of ψ .

For a given $\alpha \in]0, 1]$, if

$$\mathcal{D}^\alpha(\psi)(t) = \begin{cases} \frac{1}{\Gamma(1-\alpha)} \times \frac{d}{dt} \int_0^t (t-s)^{-\alpha} \psi(s) ds & \text{if } \alpha \in]0, 1[\\ \dot{\psi}(t) & \text{if } \alpha = 1 \end{cases}$$

exists for every $t \in \mathbb{R}_+$, $\mathcal{D}^\alpha(\psi)$ is the α -fractional derivative of ψ .

Remark. Consider $\alpha \in]0, 1]$ and $\psi : \mathbb{R}_+ \rightarrow \mathbb{R}$. If $l^\alpha(\psi)$ and $\mathcal{D}^\alpha(\psi)$ are both defined :

$$(l^\alpha \circ \mathcal{D}^\alpha)(\psi) = (\mathcal{D}^\alpha \circ l^\alpha)(\psi) = \psi.$$

It is also possible to show that B^H is a Volterra process (cf. [28], Section 5.1.3 and [6], Example 2) :

On one hand, let K_H^* be the operator defined on \mathcal{E} by :

$$\forall t \in [0, T], K_H^*(\mathbf{1}_{[0,t]})(s) = K_H(t, s) \mathbf{1}_{[0,t]}(s)$$

such that, for every $(s, t) \in \Delta_T$,

$$K_H(t, s) = \frac{(t-s)^{H-1/2}}{\Gamma(H+1/2)} \mathbf{F}\left(\frac{1}{2} - H, H - \frac{1}{2}, H + \frac{1}{2}, 1 - \frac{t}{s}\right) \mathbf{1}_{[0,t]}(s)$$

where, \mathbf{F} is the Gauss hyper-geometric function (cf. [6], Example 2).

Since K_H^* is an isometry between \mathcal{E} and $L^2([0, T])$, and $\bar{\mathcal{E}}^{(\cdot, \cdot)} \mathcal{H}_{B^H} = \mathcal{H}_{B^H}$; K_H^* admits a unique extension on \mathcal{H}_{B^H} (cf. [28], Section 5.1.3).

On the other hand, let \mathbf{B}^H be the isonormal Gaussian process associated to B^H as at Section 3.1. The stochastic process B defined on $[0, T]$ by

$$\forall t \in [0, T], B_t = \mathbf{B}^H[(K_H^*)^{-1}(\mathbf{1}_{[0,t]})]$$

is a Brownian motion. Then, B^H has the following integral representation :

$$\forall t \in [0, T], B_t^H = \int_0^t K_H(t, s) dB_s.$$

Remark. This representation allows us to give an explicit version of the isometry defined at equation (22), that we denote by I_H in the particular case of the fractional Brownian motion B^H :

Proposition 6.3. The operator I_H satisfies the following equalities :

$$\begin{aligned} I_H^{-1} &= (K_H^*)^{-1} \circ (\varphi_H \mathcal{D}^{H-1/2}) \circ \left(\frac{1}{\varphi_H} \times \frac{d}{dt} \right) \text{ if } H \geq 1/2 \text{ and} \\ I_H^{-1} &= (K_H^*)^{-1} \circ \left(\frac{1}{\varphi_H} \mathcal{D}^{1/2-H} \right) \circ (\varphi_H \mathcal{D}^{2H}) \text{ if } H \leq 1/2 \end{aligned}$$

where, φ_H is the function defined on \mathbb{R} by :

$$\forall y \in \mathbb{R}, \varphi_H(y) = y^{H-1/2} \mathbf{1}_{y \geq 0}.$$

In a sake of completeness :

Proof. On one hand, from L. Decreusefond and A. Ustunel [7] (see also [28], Section 5.1.3) ; for every $H \in]0, 1[$ and every $s, t \in [0, T]$,

$$(33) \quad \int_0^{t \wedge s} K_H(t, u) K_H(s, u) du = \mathbb{E}(B_t^H B_s^H).$$

Definitions of \mathbf{B}^H and K_H^* imply that equality (33) is equivalent to :

$$\int_0^s (K_H^* \mathbf{1}_{[0, t]})(u) K_H(s, u) du = \mathbb{E}[\mathbf{B}^H(\mathbf{1}_{[0, t]}) B_s^H].$$

Therefore, from the definition of I_H provided at equation (22) :

$$I_H(\mathbf{1}_{[0, t]}) = (J_H \circ K_H^*)(\mathbf{1}_{[0, t]})$$

where, J_H is the map defined on $L^2([0, T])$ by :

$$\forall \psi \in L^2([0, T]), J_H(\psi) = \int_0^\cdot \psi(u) K_H(., u) du.$$

Since $\overline{\mathcal{E}}^{(\cdot, \cdot)} \mathcal{H}_{B^H} = \mathcal{H}_{B^H}$ and, linear maps I_H and K_H^* are continuous from \mathcal{H}_{B^H} into $\mathcal{H}_{B^H}^1$ and $L^2([0, T])$ respectively ; equality $I_H = J_H \circ K_H^*$ is still true on \mathcal{H}_{B^H} .

On the other hand, $\mathbb{H} = K_H^*(\mathcal{H}_{B^H})$ is a closed subspace of $L^2([0, T])$ (cf. [28], Section 5.1.3). Since

$$K_H^* : \mathcal{H}_{B^H} \rightarrow \mathbb{H} \text{ and } I_H : \mathcal{H}_{B^H} \rightarrow \mathcal{H}_{B^H}^1$$

are invertible operators, the restriction $J_H|_{\mathbb{H}} = I_H \circ (K_H^*)^{-1}$ is invertible too. Moreover, from L. Decreusefond [6], Example 2 ; for every $\psi \in L^2([0, T])$,

$$\begin{aligned} J_H(\psi) &= \left[l^1 \circ (\varphi_H l^{H-1/2}) \right] \left(\frac{\psi}{\varphi_H} \right) \text{ if } H \geq 1/2 \text{ and} \\ J_H(\psi) &= \left[l^{2H} \circ \left(\frac{1}{\varphi_H} l^{1/2-H} \right) \right] (\varphi_H \psi) \text{ if } H \leq 1/2. \end{aligned}$$

Therefore, one can get an expression of J_H^{-1} and conclude. \square

Remarks :

- (1) Note that when $H = 1/2$, $I_H^{-1} = d/dt$ as proved at Section 3.1.
- (2) When $H \geq 1/2$, from [28], Proposition 5.2.2 ; for every $h \in \mathcal{H}_{B^H}^1$,

$$\begin{aligned} \delta_H[I_H^{-1}(h)] &= \delta_{1/2}[K_H^*[I_H^{-1}(h)]] \\ &= \delta_{1/2} \left[(\varphi_H \mathcal{D}^{H-1/2}) \circ \left(\frac{1}{\varphi_H} \times \frac{d}{dt} \right) (h) \right] \end{aligned}$$

where, δ_H and $\delta_{1/2}$ denote respectively the divergence operator associated to B^H and the divergence operator associated to the Brownian motion B involving in the representation of B^H as a Volterra process.

Precisely, since $\delta_{1/2}$ is matching with Skorohod's integral against B :

$$(34) \quad \delta_H[I_H^{-1}(h)] = \frac{1}{\Gamma(3/2-H)} \times \int_0^T \left[t^{H-1/2} \frac{d}{dt} \int_0^t (t-s)^{1/2-H} s^{1/2-H} \dot{h}_s ds \right] \delta_{1/2} B_t.$$

When $H \leq 1/2$, by following the same way :

$$(35) \quad \delta_H [I_H^{-1}(h)] = \frac{1}{\Gamma(1+H)\Gamma(1-2H)} \int_0^T \left[t^{1/2-H} \frac{d}{dt} \int_0^t (t-s)^{H-1/2} s^{H-1/2} \frac{d}{ds} \int_0^s (s-u)^{-2H} h_u du ds \right] \delta_{1/2} B_t.$$

7. AN APPLICATION IN FINANCE

In this section, we provide an application of Theorem 5.1 and Corollary 5.2 in a market defined by a SDE in which the volatility is the solution of an equation driven by a fractional Brownian motion.

Throughout this section, F takes its values in \mathbb{R}_+ .

Consider a financial market consisting of d risky assets and denote by $S^{\sigma;\mu}$ the associated prices process formally defined by :

$$(36) \quad \begin{cases} S^{\sigma;\mu} = c(\tilde{S}^{\sigma;\mu}) \\ d\tilde{S}_t^{\sigma;\mu} = b(\tilde{S}_t^{\sigma;\mu})dt + \sigma(X_t^\mu)dB_t^{H_1} \text{ with } \tilde{S}_0^{\sigma;\mu}, X_0^\mu \in \mathbb{R}^d. \\ dX_t^\mu = \mu(X_t^\mu)dB_t^{H_2} \end{cases}$$

On one hand, assume that :

Assumption 7.1. B^{H_1} and B^{H_2} are two independent d -dimensional fractional Brownian motions with independent components and respective Hurst parameters $H_1 > 1/4$ and $H_2 > 1/4$. Functions $b : \mathbb{R}^d \rightarrow \mathbb{R}^d$ and $\sigma, \mu : \mathbb{R}^d \rightarrow \mathcal{M}_d(\mathbb{R})$ satisfy assumptions 1.1 and 4.1 for $p = (1/H_1) \vee (1/H_2)$. The map $c : \mathbb{R}^d \rightarrow \mathbb{R}_+^d$ is such that $F \circ c$ satisfies Assumption 1.2.

By using Theorem 5.1, we show the existence and compute the sensitivity of

$$f_T(\sigma, \mu) = \mathbb{E}[F(S_T^{\sigma;\mu})] = \mathbb{E}[(F \circ c)(\tilde{S}_T^{\sigma;\mu})]$$

to any variation of the parameter μ .

Equation (36) has to be formally rewritten as follow :

$$dZ_t^{\sigma;\mu} = V_{b,\sigma;\mu}(Z_t^{\sigma;\mu})d\tilde{B}_t^{H_1,H_2}$$

where,

$$Z^{\sigma;\mu} = (\tilde{S}^{\sigma;\mu}, X^\mu), \tilde{B}^{H_1,H_2} = (B^{H_1,H_2}, \text{Id}_{[0,T]}), B^{H_1,H_2} = (B^{H_1}, B^{H_2})$$

and $V_{b,\sigma;\mu}$ is the collection of vector fields on $\mathbb{R}_1^d \oplus \mathbb{R}_2^d$ defined by :

$$\forall z, \beta \in \mathbb{R}_1^d \oplus \mathbb{R}_2^d, \forall \tau \in \mathbb{R}_+, V_{b,\sigma;\mu}(z) \cdot (\beta, \tau) = R_b(z)\tau + M_{\sigma;\mu}(z)\beta$$

with

$$R_b = \begin{pmatrix} b \circ \pi_{\mathbb{R}_1^d} \\ 0 \end{pmatrix} \text{ and } M_{\sigma;\mu} = \begin{pmatrix} \sigma \circ \pi_{\mathbb{R}_2^d} & 0 \\ 0 & \mu \circ \pi_{\mathbb{R}_2^d} \end{pmatrix}.$$

Proposition 7.2. Under Assumption 7.1, $f_T(\sigma, \cdot)$ is differentiable at point μ and for every $\tilde{\mu} \in \Sigma$, there exists two d -dimensional stochastic processes $h^{\sigma;\mu,\tilde{\mu}}$ and $\tilde{h}^{\sigma;\mu,\tilde{\mu}}$ defined on $[0, T]$ such that :

$$\partial_\mu f_T(\sigma, \mu) \cdot \tilde{\mu} = \mathbb{E} \left[\langle \mathbf{D}(F \circ S_T^{\sigma;\mu}); (I_{H_1}^{-1}(h^{\sigma;\mu,\tilde{\mu}}), I_{H_2}^{-1}(\tilde{h}^{\sigma;\mu,\tilde{\mu}})) \rangle \right]$$

with notations of Section 6.

Proof. On one hand, by construction, B^{H_1, H_2} satisfies Assumption 3.5. Then, a Gaussian geometric p -rough path \mathbb{B}^{H_1, H_2} exists over it from [12], Theorem 15.33 by taking $p = (1/H_1) \vee (1/H_2)$. Moreover, since b, σ, μ and their derivatives up to the level $[p] + 1$ are bounded, $V_{b, \sigma; \mu}$ is a collection of γ -Lipschitz vector fields for $\gamma > p$. Therefore, by Proposition 2.9, equation (36) admits a unique solution in rough paths sense :

$$Z^{\sigma; \mu} = \pi_{V_{b, \sigma; \mu}}(0, Z_0; \tilde{\mathbb{B}}^{H_1, H_2}) \text{ where } \tilde{\mathbb{B}}^{H_1, H_2} = S_{[p]}(\mathbb{B}^{H_1, H_2} \oplus \text{Id}_{[0, T]}).$$

On the other hand, consider $\tilde{\mu} \in \Sigma$ and

$$M_{\tilde{\mu}} = \begin{pmatrix} 0 & 0 \\ 0 & \tilde{\mu} \circ \pi_{\mathbb{R}_1^d} \end{pmatrix}.$$

Since B^{H_1} and B^{H_2} are two independent fractional Brownian motions with independent components, B^{H_1, H_2} satisfies assumptions 3.6 and 4.2. Therefore, from Theorem 5.1, there exists a $\mathcal{H}_{B^{H_1, H_2}}^1$ -valued random variable $\eta^{\sigma; \mu, \tilde{\mu}}$ such that :

$$\begin{aligned} \partial_\mu f_T(\sigma, \mu) \cdot \tilde{\mu} &= \partial_{M_{\sigma; \mu}} \mathbb{E}[(F \circ c \circ \pi_{\mathbb{R}_1^d})(Z_T^{\sigma; \mu})] \cdot M_{\tilde{\mu}} \\ &= \mathbb{E}[\langle \mathbf{D}(F \circ S_T^{\sigma; \mu}), I^{-1}(\eta^{\sigma; \mu, \tilde{\mu}}) \rangle]. \end{aligned}$$

Precisely, since for every $z \in \mathbb{R}_1^d \oplus \mathbb{R}_2^d$, $M_{\sigma; \mu}(z)$ is a non singular matrix by construction ; for every $t \in [0, T]$,

$$\eta_t^{\sigma; \mu, \tilde{\mu}} = \frac{1}{T} \int_0^t M_{\sigma; \mu}^{-1}(Z_s^{\sigma; \mu}) J_{s \leftarrow T}^{\tilde{\mathbb{B}}^{H_1, H_2}} \partial_{M_{\sigma; \mu}} Z_T^{\sigma; \mu} \cdot M_{\tilde{\mu}} ds.$$

Finally, since $\mathcal{H}_{B^{H_1, H_2}}^1 = \mathcal{H}_{B^{H_1}}^1 \oplus \mathcal{H}_{B^{H_2}}^1$, with notations of Section 6 :

$$I^{-1}(\eta^{\sigma; \mu, \tilde{\mu}}) = (I_{H_1}^{-1}(h^{\sigma; \mu, \tilde{\mu}}), I_{H_2}^{-1}(\tilde{h}^{\sigma; \mu, \tilde{\mu}}))$$

where, $h^{\sigma; \mu, \tilde{\mu}}$ (resp. $\tilde{h}^{\sigma; \mu, \tilde{\mu}}$) is the canonical projection of $\eta^{\sigma; \mu, \tilde{\mu}}$ on the Cameron-Martin's space of B^{H_1} (resp. B^{H_2}). \square

On the other hand, assume that :

Assumption 7.3. B^{H_1} and B^{H_2} are two independent d -dimensional fractional Brownian motions with independent components and respective Hurst parameters $H_1 > 1/2$ and $H_2 > 1/2$. Functions $b : \mathbb{R}^d \rightarrow \mathbb{R}^d$ and $\sigma, \mu : \mathbb{R}^d \rightarrow \mathcal{M}_d(\mathbb{R})$ satisfy assumptions 1.1 and 4.1 for $p+1$, with $p = (1/H_1) \vee (1/H_2)$. The map $c : \mathbb{R}^d \rightarrow \mathbb{R}_+^d$ is such that $F \circ c$ satisfies Assumption 1.2 or Assumption 1.3.

Corollary 7.4. Under Assumption 7.3, with notations of Section 6 :

$$I_{H_1}^{-1}(h^{\sigma; \mu, \tilde{\mu}}) \in \text{dom}(\delta_{H_1}) \text{ and } I_{H_2}^{-1}(\tilde{h}^{\sigma; \mu, \tilde{\mu}}) \in \text{dom}(\delta_{H_2}).$$

Moreover,

$$\partial_\mu f_T(\sigma, \mu) \cdot \tilde{\mu} = \mathbb{E} \left[F(S_T^{\sigma; \mu}) \left[\delta_{H_1}[I_{H_1}^{-1}(h^{\sigma; \mu, \tilde{\mu}})] + \delta_{H_2}[I_{H_2}^{-1}(\tilde{h}^{\sigma; \mu, \tilde{\mu}})] \right] \right].$$

Proof. It is an immediate consequence of Proposition 7.2 and Corollary 5.2. \square

8. NUMERICAL SIMULATIONS

In this section, we simulate the two sensitivities studied throughout this paper, when the driving signal of equation (1) is a fractional Brownian motion B^H with Hurst parameter $H > 1/2$ and $d = 1$.

In the sequel, we suppose that $T = 1$ and $[0, 1]$ is dissected in $N_1 = 2^{N_2}$ intervals of constant lengths $1/2^{N_2}$ (dyadic subdivision of order $N_2 \in \mathbb{N}^*$). That subdivision is denoted by $(t_k; k = 0, \dots, N_1)$. In simulations, we get discrete samples of B^H on it by using Wood-Chang's algorithm (cf. [9], Section 2.1.3).

8.1. Preliminaries. Consider $d = 1$, $\sigma, \tilde{\sigma} \in \Sigma$, $x \in \mathbb{R}$ and the three following SDEs in Young's sense :

$$(37) \quad dX_t^{x,\sigma} = b(X_t^{x,\sigma}) dt + \sigma(X_t^{x,\sigma}) dB_t^H \text{ with } X_0^{x,\sigma} = x,$$

$$(38) \quad dY_t^x = \dot{b}(X_t^{x,\sigma}) Y_t^x dt + \dot{\sigma}(X_t^{x,\sigma}) Y_t^x dB_t^H \text{ with } Y_0^x = 1 \text{ and}$$

$$(39) \quad dZ_t^{\sigma,\tilde{\sigma}} = \dot{b}(X_t^{x,\sigma}) Z_t^{\sigma,\tilde{\sigma}} dt + \dot{\sigma}(X_t^{x,\sigma}) Z_t^{\sigma,\tilde{\sigma}} dB_t^H + \tilde{\sigma}(X_t^{x,\sigma}) dB_t^H$$

with $Z_0^{\sigma,\tilde{\sigma}} = 0$.

Since Russo-Vallois integral is matching with Young's integral for $H > 1/2$, classical Euler schemes for (37), (38) and (39) with step-size N_1^{-1} are respectively given by :

$$\begin{cases} X_0^{N_1} = x \\ X_{t_k}^{N_1} = X_{t_{k-1}}^{N_1} + b(X_{t_{k-1}}^{N_1}) N_1^{-1} + \sigma(X_{t_{k-1}}^{N_1}) (B_{t_k}^H - B_{t_{k-1}}^H) \end{cases}, \\ \begin{cases} Y_0^{N_1} = 1 \\ Y_{t_k}^{N_1} = Y_{t_{k-1}}^{N_1} + \dot{b}(X_{t_{k-1}}^{N_1}) Y_{t_{k-1}}^{N_1} N_1^{-1} + \dot{\sigma}(X_{t_{k-1}}^{N_1}) Y_{t_{k-1}}^{N_1} (B_{t_k}^H - B_{t_{k-1}}^H) \end{cases} \text{ and} \\ \begin{cases} Z_0^{N_1} = 0 \\ Z_{t_k}^{N_1} = Z_{t_{k-1}}^{N_1} + \dot{b}(X_{t_{k-1}}^{N_1}) Z_{t_{k-1}}^{N_1} N_1^{-1} + \dot{\sigma}(X_{t_{k-1}}^{N_1}) Z_{t_{k-1}}^{N_1} (B_{t_k}^H - B_{t_{k-1}}^H) + \\ \tilde{\sigma}(X_{t_{k-1}}^{N_1}) (B_{t_k}^H - B_{t_{k-1}}^H) \end{cases}$$

for $k = 1, \dots, N_1$.

In [21], A. Lejay proved the following result (cf. [21], Proposition 5) :

Proposition 8.1. *Consider a continuous function $w : [0, T] \rightarrow \mathbb{R}^d$ of finite p -variation ($p \in [1, 2[$) and V a collection of differentiable vector fields on \mathbb{R}^d with a γ -Hölder continuous derivative ($\gamma \in]0, 1[$ and $\gamma + 1 > p$). Then, there exists a constant $C(T, V, w) > 0$, not depending on N_1 , such that :*

$$\|y^{N_1} - y\|_{\infty;T} \leq C(T, V, w) N_1^{1-2/p}$$

where, $dy_t = V(y_t)dw_t$ with initial condition $y_0 \in \mathbb{R}^d$ and, y^{N_1} is the associated Euler scheme with step-size $1/N_1$.

On one hand, by reading carefully the proof of Proposition 8.1 in [21] and Fernique's theorem, one can show that the random variable $C(T, V, B^H)$ belongs to $L^r(\Omega; \mathbb{P})$ for every $r > 0$. Moreover, $b, \sigma \in C^2(\mathbb{R})$ and B^H has α -Hölder continuous paths with $\alpha \in]1/2, H]$. Therefore, by Proposition 8.1 :

$$\forall r > 0, \lim_{N_1 \rightarrow \infty} \mathbb{E} \left(\|X^{N_1} - X^{x,\sigma}\|_{\infty;1}^r \right) = 0.$$

On the other hand, equations (38) and (39) can be rewritten as follow :

$$dY_t^x = A(Y_t^x) dB_t^{x,\sigma,H} \text{ and } dZ_t^{\sigma,\tilde{\sigma}} = \tilde{A}(Z_t^{\sigma,\tilde{\sigma}}) \cdot (dB_t^{x,\sigma,H}, d\tilde{B}_t^{x,\sigma,H})$$

where,

$$dB_t^{x,\sigma,H} = \dot{b}(X_t^{x,\sigma}) dt + \dot{\sigma}(X_t^{x,\sigma}) dB_t^H \text{ and } d\tilde{B}_t^{x,\sigma,H} = \tilde{\sigma}(X_t^{x,\sigma}) dB_t^H$$

and, A and \tilde{A} are two linear vector fields defined on \mathbb{R} by $A(y) \cdot w = yw$ and $\tilde{A}(y) \cdot (w, v) = yw + v$ for every $v, w, y \in \mathbb{R}$.

Since B^H and then, $X^{x,\sigma}$ have α -Hölder continuous paths with $\alpha \in]1/2, H]$, $B^{x,\sigma,H}$

has also α -Hölder continuous paths from elementary properties of Young's integral (cf. [12], Theorem 6.8). Therefore, since A and \tilde{A} are linear vector fields, assumptions of Proposition 8.1 are satisfied :

$$\forall r > 0, \lim_{N_1 \rightarrow \infty} \mathbb{E} \left(\|Y^{N_1} - Y^x\|_{\infty;1}^r \right) = 0 \text{ and } \lim_{N_1 \rightarrow \infty} \mathbb{E} \left(\|Z^{N_1} - Z^{\sigma, \tilde{\sigma}}\|_{\infty;1}^r \right) = 0$$

because, $C(T, A, B^{x, \sigma, H})$ and $C[T, \tilde{A}, (B^{x, \sigma, H}, \tilde{B}^{x, \sigma, H})]$ belong to $L^r(\Omega; \mathbb{P})$.

Remark. Note that from I. Nourdin and A. Neuenkirch [27], Theorem 1 :

$$N_1^{2H-1} (X_1^{N_1} - X_1^{x, \sigma}) \xrightarrow[N_1 \rightarrow \infty]{\text{a.s.}} -\frac{1}{2} \int_0^1 \dot{\sigma}(X_s^{x, \sigma}) \mathbf{D}_s X_1^{x, \sigma} ds.$$

That result is older than [21], Proposition 5.

8.2. Simulations for F differentiable. First, let's provide two converging estimators :

Proposition 8.2. *Consider :*

$$\begin{aligned} \Theta_{n, N_1}^x &= \frac{1}{n} \sum_{i=1}^n \dot{F}(X_1^{i, N_1}) Y_1^{i, N_1}, \quad \theta^{x, N_1} = \mathbb{E} \left[\dot{F}(X_1^{N_1}) Y_1^{N_1} \right] \text{ and} \\ \Theta_{n, N_1}^{\sigma, \tilde{\sigma}} &= \frac{1}{n} \sum_{i=1}^n \dot{F}(X_1^{i, N_1}) Z_1^{i, N_1}, \quad \theta^{\sigma, \tilde{\sigma}, N_1} = \mathbb{E} \left[\dot{F}(X_1^{N_1}) Z_1^{N_1} \right] \end{aligned}$$

where,

$$(X^{1, N_1}, Y^{1, N_1}, Z^{1, N_1}), \dots, (X^{n, N_1}, Y^{n, N_1}, Z^{n, N_1})$$

are $n \in \mathbb{N}^*$ independent copies of $(X^{N_1}, Y^{N_1}, Z^{N_1})$.

On one hand, under Assumption 1.2 :

$$(40) \quad \Theta_{n, N_1}^x \xrightarrow[n \rightarrow \infty]{\mathbb{P}} \theta^{x, N_1} \xrightarrow[N_1 \rightarrow \infty]{} \partial_x f_T(x, \sigma) \text{ and}$$

$$(41) \quad \Theta_{n, N_1}^{\sigma, \tilde{\sigma}} \xrightarrow[n \rightarrow \infty]{\mathbb{P}} \theta^{\sigma, \tilde{\sigma}, N_1} \xrightarrow[N_1 \rightarrow \infty]{} \partial_\sigma f_T(x, \sigma) \cdot \tilde{\sigma}.$$

On the other hand,

$$(42) \quad \sqrt{n} \frac{\Theta_{n, N_1}^x - \theta^{x, N_1}}{\hat{s}_{n, N_1}^x} \xrightarrow[n \rightarrow \infty]{\mathcal{D}} \mathcal{N}(0, 1) \text{ and}$$

$$(43) \quad \sqrt{n} \frac{\Theta_{n, N_1}^{\sigma, \tilde{\sigma}} - \theta^{\sigma, \tilde{\sigma}, N_1}}{\hat{s}_{n, N_1}^{\sigma, \tilde{\sigma}}} \xrightarrow[n \rightarrow \infty]{\mathcal{D}} \mathcal{N}(0, 1)$$

where, \hat{s}_{n, N_1}^x and $\hat{s}_{n, N_1}^{\sigma, \tilde{\sigma}}$ are the empirical standard deviations of

$$\begin{aligned} &\dot{F}(X_1^{1, N_1}) Y_1^{1, N_1}, \dots, \dot{F}(X_1^{n, N_1}) Y_1^{n, N_1} \text{ and} \\ &\dot{F}(X_1^{1, N_1}) Z_1^{1, N_1}, \dots, \dot{F}(X_1^{n, N_1}) Z_1^{n, N_1} \end{aligned}$$

respectively.

Proof. Under Assumption 1.2, by preliminaries ; for every $r > 0$,

$$\dot{F}(X_1^{N_1}) Y_1^{N_1} \xrightarrow[N_1 \rightarrow \infty]{L^r} \dot{F}(X_1^{x, \sigma}) Y_1^x \text{ and } \dot{F}(X_1^{N_1}) Z_1^{N_1} \xrightarrow[N_1 \rightarrow \infty]{L^r} \dot{F}(X_1^{x, \sigma}) Z_1^{\sigma, \tilde{\sigma}}.$$

Therefore, (40) and (41) are true by the law of large numbers and, (42) and (43) are true by the central limit theorem together with Slutsky's lemma. \square

Via the second part of Proposition 8.2, we obtain the two following α -confidence intervals ($\alpha \in]0, 1[$) :

$$\mathbb{P} \left(\Theta_{n, N_1}^x - \frac{t_\alpha}{\sqrt{n}} \hat{s}_{n, N_1}^x \leq \theta^{x, N_1} \leq \Theta_{n, N_1}^x + \frac{t_\alpha}{\sqrt{n}} \hat{s}_{n, N_1}^x \right) \simeq 1 - \alpha$$

and

$$\mathbb{P} \left(\Theta_{n, N_1}^{\sigma, \tilde{\sigma}} - \frac{t_\alpha}{\sqrt{n}} \hat{s}_{n, N_1}^{\sigma, \tilde{\sigma}} \leq \theta^{\sigma, \tilde{\sigma}, N_1} \leq \Theta_{n, N_1}^{\sigma, \tilde{\sigma}} + \frac{t_\alpha}{\sqrt{n}} \hat{s}_{n, N_1}^{\sigma, \tilde{\sigma}} \right) \simeq 1 - \alpha$$

where, $\Phi(t_\alpha) = 1 - \alpha/2$ and Φ is the repartition function of $\mathcal{N}(0, 1)$.

Numerical application. Suppose that $H = 0.6$, $N_1 = 2^{N_2}$ with $N_2 = 15$ and $n = 500$. Moreover, suppose that for every $y \in \mathbb{R}$, $b(y) = 0$, $\sigma(y) = 1 + e^{-y^2}$, $\tilde{\sigma}(y) = 1 + \pi/2 + \arctan(y)$, $F(y) = y^2$ and $x = 1$:

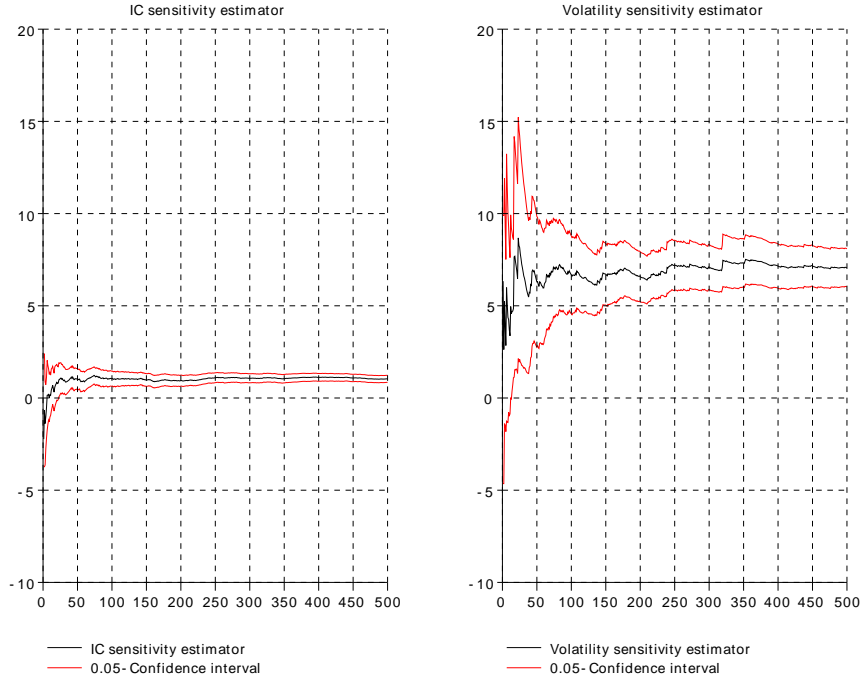


FIGURE 1. Convergence of estimators

These are representations of

$$i \in \{1, \dots, n\} \mapsto \Theta_{i, N_1}^x(\omega) \text{ and } i \in \{1, \dots, n\} \mapsto \Theta_{i, N_1}^{\sigma, \tilde{\sigma}}(\omega)$$

for a given $\omega \in \Omega$ and then evaluate the convergence of estimators. Points of lateral curves are bounds of the 0.05-confidence intervals at steps $i = 1, \dots, n$ for each estimator. Note that Θ^x seems to converge faster than $\Theta^{\sigma, \tilde{\sigma}}$. Precisely :

Statistics	Values
$\Theta_{n,N_1}^x(\omega)$	1.042
0.05-confidence interval	[0.851; 1.232]
CI's length	0.381
$\Theta_{n,N_1}^{\sigma,\tilde{\sigma}}(\omega)$	7.112
0.05-confidence interval	[6.071; 8.154]
CI's length	2.083

Confidence intervals lengths confirm that Θ^x converges faster than $\Theta^{\sigma,\tilde{\sigma}}$.

REFERENCES

- [1] E. Benhamou. *Optimal Malliavin Weighting Function for the Computation of the Greeks*. Mathematical Finance, Volume 13, Issue 1, 2003.
- [2] T. Cass, C. Litterer and T. Lyons. *Integrability Estimates for Gaussian Rough Differential Equations*. arXiv:1104.1813v3, 2011.
- [3] L. Coutin. *Rough Paths via Sewing Lemma*. ESAIM : Probability and Statistics, doi:10.1051/ps/2011108.
- [4] L. Coutin, P. Friz and N. Victoir. *Good Rough Paths Sequences and Applications to Anticipating and Fractional Stochastic Calculus*. Ann. Probab., 35(3) : 1172-1193, 2007.
- [5] A.M. Davie. *Differential Equations Driven by Rough Paths : An Approach via Discrete Approximation*. Appl. Math. Res. Express AMRX, (2) : Art. ID abm009, 40, 2007.
- [6] L. Decreusefond. *Stochastic Integration with Respect to Volterra Processes*. Ann. Inst. H. Poincaré Probab. Statist., 41(2):123-149, 2005.
- [7] L. Decreusefond and A. Ustunel. *Stochastic Analysis of the Fractional Brownian Motion*. Potential Anal., 10(2) : 177-214, 1999.
- [8] L. Decreusefond and N. Lavad. *Simulation of the Fractional Brownian Motion and Application to the Fluid Queue*. Proceeding of the ATNAC '96 conference, 1996.
- [9] T. Dieker. *Simulation of Fractional Brownian Motion*. Master thesis, University of Twente, 2004.
- [10] E. Fournié, J-M. Lasry, J. Lebuchoux, P-L. Lions and N. Touzi. *Applications of Malliavin Calculus to Monte-Carlo Methods in Finance*. Finance Stochast. 3, 391-412, 1999.
- [11] P. Friz and N. Victoir. *Differential Equations Driven by Gaussian Signals*. Ann. Inst. Henri Poincaré Probab. Stat. 46, no. 2, 369-411, 2010.
- [12] P. Friz and N. Victoir. *Multidimensional Stochastic Processes as Rough Paths : Theory and Applications*. Cambridge Studies in Applied Mathematics, 120. Cambridge University Press, Cambridge, 2010.
- [13] P. Friz and S. Riedel. *Integrability of Linear Rough Differential Equations*. arXiv:1104.0577, 2011.
- [14] G. Bernis, E. Gobet and A. Kohatsu-Higa. *Monte Carlo Evaluation of Greeks for Multidimensional Barrier and Lookback Options*. Mathematical Finance, 13(1):99-113, 2003.
- [15] E. Gobet and R. Müños. *Sensitivity Analysis using Itô-Malliavin Calculus and Martingales, and Application to Stochastic Optimal Control*. Siam J. Control Optim. Vol. 43, No. 5, pp. 1676-1713, 2005.
- [16] Y. Hu, B. Oksendal, T. Zhang and F. Biagini. *Stochastic Calculus for Fractional Brownian Motion and Applications*. Probability and its Applications (New York), Springer-Verlag, Berlin, 2009.
- [17] H.T. Huynh, V.S. Lai and I. Soumaré. *Simulations stochastiques et applications en finance avec programmes Matlab*. Collection Finance, Economica, 2006.
- [18] Y. El Khatib and N. Privault. *Computations of Greeks in Markets with Jumps via the Malliavin Calculus*. Finance and Stochastics 8, 161-179, 2004.
- [19] H. Kunita. *Stochastic Flows and Stochastic Differential Equations*. Cambridge Studies in Applied Mathematics, 24. Cambridge University Press, Cambridge, 1997.
- [20] D. Lamberton and B. Lapeyre. *Introduction au calcul stochastique appliqué à la finance. Seconde Edition*. Mathématiques et Applications, Ellipses, 1997.
- [21] A. Lejay. *Controlled Differential Equations as Young Integrals : A Simple Approach*. Journal of Differential Equations 248, 1777-1798, 2010.
- [22] T. Lyons. *Differential Equations Driven by Rough Signals*. Rev. Mat. Iberoamericana, 14(2):215-310, 1998.
- [23] T. Lyons and Z. Qian. *System Control and Rough Paths*. Oxford University Press, 2002.
- [24] P. Malliavin and A. Thalmaier. *Stochastic Calculus of Variations in Mathematical Finance*. Springer Finance, Springer-Verlag, Berlin, 2006.

- [25] J. Neveu. *Processus aléatoires gaussiens*. Presses de l'université de Montréal, 1968.
- [26] R. Norberg. *Sensitivity Analysis in Insurance and Finance*. Information processes, Kalashnikov Memorial Seminar 2, pp. 240-242, 2002.
- [27] I. Nourdin and A. Neuenkirch. *Exact Rate of Convergence of some Approximation Schemes Associated to SDEs Driven by a Fractional Brownian Motion*. Journal of Theoretical Probability, 20(4), pp. 871-899, 2007.
- [28] D. Nualart. *The Malliavin Calculus and Related Topics. Second Edition*. Probability and its Applications (New York), Springer-Verlag, Berlin, 2006.
- [29] C. Pintoux. *Calculs stochastique et de Malliavin appliqués aux modèles de taux d'intérêt engendrant des formules fermées*. tel-00555727, version 1, 2011.
- [30] V.V. Piterbarg. *Risk Sensitivities of Bermuda Options*. Technical report, Bank of America, 2002.
- [31] N. Privault and X. Wei. *A Malliavin Calculus Approach to Sensitivity in Insurance*. Insurance : Mathematics and Economics 35, 679-690, 2004.
- [32] M.I. Reiman and A. Weiss. *Sensitivity Analysis via Likelihood Ratios*. In J. Wilson, J. Henriksen and S. Roberts, Editors, Proceedings of the 1986 Winter Simulation Conference, pages 285-289, 1986.
- [33] D. Revuz and M. Yor. *Continuous Martingale and Brownian Motion. Third Edition*. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Vol. 293, Springer-Verlag, Berlin, 1999.
- [34] L.C.G. Rogers. *Arbitrage with Fractional Brownian Motion*. Mathematical Finance, Vol. 7, No. 1, 95-105, 1997.
- [35] S. Samko, A. Kilbas, O. Marichev. *Fractional Integrals and Derivatives*. Gordon and Breach Science, 1993.
- [36] J. Teichmann and N. Touzi. *Calculating the Greeks for Hypoelliptic Problems by Universal Weights*. Working Paper, 2004.
- [37] J. Teichmann. *Calculating the Greeks by Cubature Formulas*. Proceedings of the Royal Society London A 462, 647-670, 2006.
- [38] J. Yang and H. J. Kushner. *A Monte Carlo Method for Sensitivity Analysis and Parametric Optimization of Nonlinear Stochastic Systems*. SIAM J. Control Optim. 29, pp. 1216-1249, 1991.
- [39] V. Vovk. *Rough Paths in Idealized Financial Markets*. Lithuanian Mathematical Journal, 2010.

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